THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1
Solution to Homework 6

1. If X and Y are Banach spaces and $T_n\colon X \to Y$, $n = 1, 2, \dots$ a sequence of bounded linear operators, show that the following statements are equivalent:

   (a) the sequence $(\|T_n\|)$ is bounded,

   (b) the sequence $(\|T_n x\|)$ is bounded for every $x \in X$,

   (c) the sequence $(|f(T_n x)|)$ is bounded for every $x \in X$ and every $f \in Y^*$.

   Proof. We prove in the order (a) $\implies$ (b) $\implies$ (c) $\implies$ (a).

   (a) $\implies$ (b) There exists $M > 0$ such that $\sup_n \|T_n\| \le M$. Fix any $x \in X$. Then for all $n \in \mathbb{N}$,
   $$\|T_n x\| \le \|T_n\| \|x\| \le M\|x\| < \infty.$$

   (b) $\implies$ (c) Fix any $x \in X$, there exists $M_x > 0$ such that $\sup_n \|T_n x\| \le M_x$. Fix any $f \in Y^*$. Then for all $n \in \mathbb{N}$,
   $$\|f(T_n x)\| \le \|f\| \|T_n x\| \le \|f\| M_x < \infty.$$

   (c) $\implies$ (a) Let $Q\colon Y \to Y^{**}$ be the canonical mapping. Fix any $x \in X$. Since Y^* is a Banach space and for every $f \in Y^*$, by (c) we have
   $$\|Q(T_n x)(f)\| = \|f(T_n x)\| < \infty.$$
   By Uniform Boundedness Theorem there exists $M_x > 0$ (independent of f) such that for all $n \in \mathbb{N}$,
   $$\|T_n x\| = \|Q(T_n x)\| \le M_x < \infty.$$
   Since X is a Banach space and the above inequality holds from all $x \in X$, by Uniform Boundedness Theorem there exists $M > 0$ (independent of x) such that $\|T_n\| \le M$ for all $n \in \mathbb{N}$.

   □

2. Let X and Y be normed spaces and $T\colon X \to Y$ a closed linear operator (the graph of T is closed).

   (a) Show that the image of a compact subset of X is closed in Y.

   (b) Show that the inverse image of a compact subset of Y is closed in X.

   Proof. Note that the closedness of T means
   $$\begin{cases} x_n \to x \in X \\ Tx_n \to y \in Y \end{cases} \implies Tx = y. \tag{1}$$

(a) Let K be a compact subset of X. Suppose otherwise that TK is not closed. Then since Y is a metric space, there exists $y \in Y \setminus TK$ such that $Tx_n \to y$ for a sequence (x_n) in K. Since K is compact and X is a metric space, then K is sequentially compact. Hence by passing to a subsequence we may assume $x_n \to x$ for some $x \in K$. This implies

$$\begin{cases} x_n \to x \\ Tx_n \to y \end{cases} \quad \text{but } y \neq Tx,$$

which contradicts (1).

(b) Let K be a compact subset of Y. Suppose otherwise that $T^{-1}K$ is not closed. Then since X is a metric space, there exists $x \in X \setminus T^{-1}K$ such that $x_n \to x$ for a sequence (x_n) in $T^{-1}K$. Since K is compact and Y is a metric space, then K is sequentially compact. Hence by passing to a subsequence we may assume $Tx_n \to y$ for some $y \in K$. This implies

$$\begin{cases} x_n \to x \\ Tx_n \to y \end{cases} \quad \text{but } y \neq Tx,$$

which contradicts (1).

□

— *THE END* —