THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1
Solution to Homework 3

1. Show that $p(x) = \limsup x(n)$, where $x = (x(n)) \in \ell_\infty$, $x(n) \in \mathbb{R}$, defines a sublinear functional on ℓ_∞.

Proof. Since $(x(n))$ is bounded for $(x(n)) \in \ell_\infty$, we have $p(x)$ is well-defined on ℓ_∞.

(sub-additive) For $y = (y(n)), x = (x(n)) \in \ell_\infty$,

$$\begin{aligned} p(x+y) &= \limsup(x(n)+y(n)) = \lim_{n\to\infty} \sup_{k\geq n}(x(k)+y(k)) \leq \lim_{n\to\infty} (\sup_{k\geq n} x(k) + \sup_{k\geq n} y(k)) \\ &= \limsup x(n) + \limsup y(n) = p(x) + p(y). \end{aligned}$$

(positive homogeneous) Let $\alpha \geq 0$ and $x \in \ell_\infty$. Then

$$p(\alpha x) = \limsup(\alpha x(n)) = \lim_{n\to\infty} \sup_{k\geq n} \alpha x(k) = \alpha \limsup x(n) = \alpha p(x).$$

This concludes that p is a sublinear functional on ℓ_∞. □

2. Let p be the Minkowski functional for an open convex neighborhood U of 0 in a normed space X.

   (a) Show that for $x \neq 0$, $p(x) = 0$ if and only if $x \in tU$ for every $t > 0$, so U is "unbounded in the direction of the vector x."

   (b) Show that $p(x) \leq 1$ if $x \in U$, and $p(x) \geq 1$ if $x \notin U$.

Proof. Since U is convex and $0 \in U$, we have $\lambda U = \lambda U + (1-\lambda)\cdot 0 \subset U$ for all $\lambda \in [0,1]$. In particular, for $0 \leq t \leq s$, if $x \in tU$ for some $x \in X$, then $x \in tU = s((t/s)U) \subset sU$ since $t/s \leq 1$. Recall

$$p(x) := \inf\{t > 0 \colon x \in tU\}. \tag{1}$$

(a) Let $x \in X \setminus \{0\}$. If $p(x) = 0$, then for any $t > 0$, by (1) there exists $\varepsilon < t$ such that $x \in \varepsilon U$, thus $x \in tU$ by the previous argument. On the other hand, by (1) we have $p(x) = 0$ if $x \in tU$ for all $t > 0$.

(b) If $x \in U = 1 \cdot U$, then $p(x) \leq 1$ by (1).

Let $x \notin U$. Suppose otherwise that $p(x) < 1$, then $x \in tU$ for some $t \in (0,1)$. Thus $x \in tU \subset U$ by the previous argument, which contradicts $x \notin U$. Hence $p(x) \geq 1$.

□

— THE END —