THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1

Solution to Homework 2

1. (*Bounded linear extension theorem*) Let X be a normed space and $\widetilde{X}$ its Banach completion. If f is a bounded linear functional on X, then there exists a unique linear functional $\widetilde{f}$ on $\widetilde{X}$ such that $\widetilde{f}|_X = f$ and $\|\widetilde{f}\| = \|f\|$.

Proof. Denote the scalar field by $\mathbf{F}$. Let $\iota\colon X \to \widetilde{X}$ be the isometric embedding such that $\iota(X)$ is dense in $\widetilde{X}$. For notational brevity, we treat X as a subset of $\widetilde{X}$ and write $\iota(x)$ as x.

Let $x \in \widetilde{X}$. Since $\widetilde{X}$ is a metric space and $\overline{X} = \widetilde{X}$, there exist (x_n) in X such that $x = \lim_{n\to\infty} x_n$ in norm $\|\cdot\|$. Since $|f(x_m) - f(x_n)| \le \|f\|\|x_m - x_n\|$, we have $f(x_n)$ is also a Cauchy sequence, and so $\lim_{n\to\infty} f(x_n)$ exists by the completeness of $\mathbf{F}$. Then for $x \in \widetilde{X}$, we can define

$$\widetilde{f}(x) := \lim_{n\to\infty} f(x_n) \tag{1}$$

where (x_n) is any sequence in X such that $x = \lim_{n\to\infty} x_n$.

(i) (well-defined) Let $(x_n), (y_n)$ be any two sequences in X such that $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. By triangle inequality,

$$|f(x_n) - f(y_n)| \le \|f\|\|x_n - y_n\| \le \|f\|\left(\|x - x_n\| + \|x - y_n\|\right) \to 0$$

as $n \to \infty$. Hence $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$.

(ii) (linear) Let $\alpha \in \mathbf{F}$ and $x, y \in \widetilde{X}$ with $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$, where $(x_n), (y_n)$ are in X. Then $\alpha x + y = \lim_{n\to\infty}(\alpha x_n + y_n)$ and

$$\begin{aligned}\widetilde{f}(\alpha x + y) &= \lim_{n\to\infty} f(\alpha x_n + y_n) = \lim_{n\to\infty} (\alpha f(x_n) + f(y_n)) \\ &= \alpha \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} f(y_n) = \alpha\widetilde{f}(x) + \widetilde{f}(y),\end{aligned}$$

where second equality is by the linearity of f and the third equality is by the continuity of scalar product and addition.

(iii) (extension) Let $x \in X$. By taking the constant sequence $(x)_{n=1}^{\infty}$, we have $\widetilde{f}(x) = \lim_{n\to\infty} f(x) = f(x)$. Hence $\widetilde{f}|_X = f$.

(iv) (bounded with equal norm) For $x \in \widetilde{X}$, it follows from (1) that

$$|\widetilde{f}(x)| = \lim_{n\to\infty} |f(x_n)| \le \lim_{n\to\infty} \|f\|\|x_n\| = \|f\|\|x\|$$

since $x_n \xrightarrow{\|\cdot\|} x$. Hence $\|\widetilde{f}\| \le \|f\|$. On the other hand, since $X \subset \widetilde{X}$ and $\widetilde{f}|_X = f$, we have $\|\widetilde{f}\| \ge \|f\|$. Together we have $\|\widetilde{f}\| = \|f\|$.

(v) (unique) Let $\widetilde{g} \in \widetilde{X}^*$ such that $g|_X = f$ and $\|g\| = \|f\|$. Let $x \in \widetilde{X}$ and take a sequence (x_n) in X such that $x = \lim_{n\to\infty} x_n$. By the continuity of g and $g(x_n) = f(x_n)$,

$$g(x) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = \widetilde{f}(x),$$

thus $g = \widetilde{f}$ on $\widetilde{X}$.

□

2. Let X, Y and Z be normed spaces and $S\colon X \to Y$ and $T\colon Y \to Z$ bounded operators. Prove that

$$\|TS\| \le \|T\|\|S\|.$$

Proof. Let $T\colon X \to Y$ be any bounded operator. By $\|T\| := \sup_{x\neq 0} \|Tx\|/\|x\| < \infty$,

$$\|Tx\| \le \|T\|\|x\| \tag{2}$$

for all $x \in X \setminus \{0\}$. We show that $T0 = 0$ (without using the linearity). Suppose otherwise that $\|T0\| > 0$. Then by the triangle inequality and (2),

$$\|T\| \ge \frac{\|Tx_n\|}{\|x_n\|} \ge \frac{\|T0\| - \|Tx_n\|}{\|x_n\|} \ge \frac{\|T0\| - \|T\|\|x_n\|}{\|x_n\|} = \frac{\|T0\|}{\|x_n\|} - \|T\| \to \infty \text{ as } n \to \infty$$

for any nonzero sequence $x_n \to 0$, which contradicts $\|T\| < \infty$. Thus (2) holds for all $x \in X$. Applying (2) to bounded operators T and S in the assumption shows that

$$\|TSx\| \le \|T\|\|Sx\| \le \|T\|\|S\|\|x\|$$

for all $x \in X$. Hence $\|TS\| \le \|T\|\|S\|$ by the definition of $\|TS\|$. □

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