MATH 3060 Tutorial 8

Chan Ki Fung

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1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ with Df non-degenerate everywhere.
   (a) Does f always send open sets to open sets?
   (b) Does f always send closed sets to closed sets?

2. Thinking $\mathbb{R}^{n\times n}$ as the set of $n \times n$ matrices. Let $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be $T(A) = A^2$, show that DT is non-degenerate at I. Hence show that there exists a $\delta > 0$, s.t. for $||B|| < \delta$, $I + B$ has a square root.

3. In the setting of theorem 3.10 and proposition 3.11 in lecture 15, we define

   $$X = \{\phi \in C[t_0 - a', t_0 + a'] : \phi(t_0) = x_0, \phi(t) \in [x_0 - b, x_0 + b]\}$$

   and a function $T : (X, d_\infty) \to (X, d_\infty)$

   $$(T\phi)(t) \coloneqq x_0 + \int_{t_0}^{t} f(t, \phi(t)))dt.$$

   (a) Show that

   $$d_\infty(T^k(\phi), T^k(\psi)) \leq \frac{L^k a'^k}{k!} d_\infty(\phi, \psi)$$

   (b) Show that as long as $a'M < b$, T has a unique fixed point.