MATH 3060 Tutorial 7

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1. Let X be a complete metric space, and $T : X \to X$ a map. Suppose T^3 is a contraction, show that T has a unique fixed point.

2. (a) Let X be a Banach space, and $Y \subset X$ be a subspace. Suppose Y is not dense in X, show that there exists $x \in X$, with $||x|| = 1$ and $d(x, Y) > \frac{1}{2}$.

   (b) Let X be a Banach space, can you find a sequence $\{x_n\} \subset X$ such that $||x_n|| = 1$ for all n, and there is no converging subsequences.

   Proof.

   (a) Since Y is not closed, we can choose $x_0 \in X$ s.t. $d(x_0, Y) = c > 0$. Now, choose $y_n \in Y$ s.t. $||x_0 - y_n|| \to c$, and take $x_n = x_0 - y_n$. We have
   $$d\left(\frac{x_n}{||x_n||}, Y\right) = \frac{d(x_n, Y)}{|x_n|} \to \frac{c}{c} = 1.$$
   Just take $x = \frac{x_n}{||x_n||}$ for some large n.

   (b) By part (a), we can choose $x_1, x_2, \ldots$ so that $||x_n|| = 1$ and
   $$d(x_n, \text{span}\{x_1, x_2, \ldots, x_{n-1}\}) > \frac{1}{2}$$
   for any n. It has no converging subsequence because $||x_i - x_j|| > \frac{1}{2}$ for any $i \neq j$.

   □

3. Show that the system
   $$\begin{cases} x - 2y^3 = 0.01 \\ y + \sin^2 x = 0 \end{cases}$$
   has a solution.

4. (An application of Inverse function theorem) Consider the function $p : \mathbb{R}^n \to \mathbb{R}^n$ given by
   $$p(x) = (p_1(x), p_2(x), \ldots, p_n(x)),$$
   where $p_k(x) = x_1^k + x_2^k + \cdots x_n^k$.

(a) Show that p is not a local diffeomorphism in a neighbourhood of any point on the plane $x_1 = x_2$

(b) Show that the Jacobian

$$J = \det\left(\frac{\partial p_i}{\partial x_j}\right) = n! \cdot \prod_{i<j}(x_i - x_j)$$

(c) What if we replace p_i by the elementary symmetric polynomials?

Proof.

(a) Since $p(x+\epsilon, x, x_3, \dots) = p(x, x+\epsilon, x_3, \dots)$, p is not injective in any neighbourhood of $(x, x, x_3, \dots)$, and in particular not local diffeomorphism near that point.

(b) part (a) tells us that J vanishies on the plane $x_1 - x_2 = 0$. On the other hand, we know that J is a polynomial, so $(x_1 - x_2)$ must be a factor of J. Similarly, $(x_i - x_j)$ is a factor of of J. Therefore, we must have

$$J = c\prod_{i<j}(x_i - x_j)$$

for some polynomial c. Now if we compare the degree of both sides, c must be a constant. By comparing the coefficents of $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$, we see that $c = n!$.

(c) Similar to (b), but $c = (-1)^{n(n-1)/2}$ this time.

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