MATH 3060 Tutorial 12

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November 30, 2022

1. Let $X = (0, \infty) \subset \mathbb{R}$. Consider the metrics $d(x, y) = |x - y|$, $\rho(x, y) = |x - y| + |\frac{1}{x} - \frac{1}{y}|$.

   (a) Show that a sequence converges in d if and only if it converges in ρ.

   (b) Is d complete? Is ρ complete?

2. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable with a differentiable inverse G, show that the Jacobian matrix of G must be nonsingular everywhere.

3. Show that the system
$$\begin{cases} x - 2y^3 = 0.01 \\ y + \sin^2 x = 0 \end{cases}$$
   has a solution.

4. Let f be a C^1 function in $\mathbb{R}^2$ satisfying $|f(x, t)| \le 1 + |x|$ for all $(t, x) \in \mathbb{R}^2$. Show that the initial value problem $x' = f(t, x), x(0) = 0$ has a solution $x(t)$ for all $t \in (-\infty, \infty)$

5. Let $\mathcal{G}$ be a precompact subset of $C[0, 1]$ and $\mathcal{K}$ be a precompact subset of $C[0, 1] \times C[0, 1]$. Define, for each $g \in \mathcal{G}$ and $K \in \mathcal{K}$, the map $T_{g,K} : C[0, 1] \to C[0, 1]$ by
$$(T_{g,K} f)(x) = \int_0^1 K(x, t) f(t) dt + g(x)$$
   for any $f \in C[0, 1]$. Show that if $\mathcal{C}$ is a bounded subset of $C[0, 1]$, then the subset
$$\cup_{g \in \mathcal{G}, K \in \mathcal{K}} T_{g,K}(\mathcal{C})$$
   is precompact in $C[0, 1]$.

6. (a) State the Baire Category theorem.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For $x \in \mathbb{R}$, we define
$$\text{osc}_f(x) = \lim_{r \to 0} \sup\{|f(y) - f(z)| : y, z \in (x - r, x + r)\}.$$
   Show that f is continuous at 0 if and only if $\text{osc}_f(x) = 0$.

(c) Show that the set of continuities of a function $f : \mathbb{R} \to \mathbb{R}$ cannot be $\mathbb{Q}$.

(d) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ whose set of discontinuities is $\mathbb{Q}$.