MATH 3060 Assignment 7 solution

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1. (a) First C_{x_0,y_0} is closed, because if $f_n \in C_{x_0,y_0}$ and $f_n \to f$ uniformly, then $f(x_0) = \lim f_n(x_0) = y_0$. C_{x_0,y_0} is nowhere dense because for $f \in C_{x_0,y_0}$, $f + c \neq C_{x_0,y_0}$ for any $c \neq 0$.

   (b) The set in the question is equal to
$$\bigcup_{n\in\mathbb{N}, y\in\mathbb{Q}} C_{\frac{1}{n},q}.$$

2. Any finite subset of $\mathbb{R}$ is nowhere dense, hence the set of real roots of a polynomial is nowhere dense in $\mathbb{R}$. The set of (real) algebraic number is equal to the union of the set of real roots of all integral polynomials. It suffices to note that the set of integral polynomials is countable, because this set is equal to
$$\bigcup_{n=0}^{\infty} \{\text{integral polynomials of degree } \leq n\} \sim \bigcup_{n=0}^{\infty} \mathbb{Z}^n.$$

3. Because $G^c = \cup G_n^c$, it suffices to show that each G_n is open dense. G_n is open by definition, G_n is dense in X because
$$\overline{G_n} \supset \overline{G} = X.$$

4. Let X be a countable complete metric space. Since $X = \cup_{p\in X}\{p\}$, we know $\overline{p}$ must have nonempty interior for some $p \in X$. But $\overline{p} = p$ (true for any metric space), so $\{p\}$ has also to be open in X. This shows p is an isolated point of X.