MATH 3060 Assignment 2 solution

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1. We use the following formula in Lecture 2:

$$a_n(f') = nb_n(f)$$

$$b_n(f') = na_n(f).$$

Apply the formula repeatedly, we have

$$|a_n(f)| \leq \frac{1}{n^k} \max\{|a_n(f^{(k)})|, |b_n(f^{(k)})|\} = o(\frac{1}{n^k})$$

The last equality is Riemann Lebesgue lemma applied to $f^{(k)}$.

2. (a) Put $\phi = f - h, \psi = h - g$, we have

$$\begin{aligned} & ||f - g||_2 \leq ||f - h||_2 + ||h - g||_2 \\ \iff & ||\phi + \psi||_2 \leq ||\phi||_2 + ||\psi||_2 \\ \iff & \int (\phi + \psi)^2 \leq \int \phi^2 + 2\sqrt{\int \phi^2 \cdot \int \psi^2} + \int \psi^2 \\ \iff & \left(\int \phi\psi\right)^2 \leq \int \phi^2 \cdot \int \psi^2 \end{aligned}$$

This is Cauchy-Schwartz inequality. If $\psi = 0$ almost everywhere, the inequality obviously hold. Otherwise, the inequality can be proved by note that for any $t \in \mathbb{R}$,

$$0 \leq \int (\phi + t\psi)^2 = \int \phi^2 + t^2 \int \psi^2 + 2t \int \phi\psi$$

putting $t = -\frac{\int \phi\psi}{\int \psi^2}$, we get the inequality. Also note that equality holds precisely when $\phi + t\psi = 0$ almost everywhere. That is when ϕ is equal to a multiple of ψ almost everywhere. (or when one of ψ vanish almost everywhere.) Geometrically, this means f, g and h are collinear (almost everywhere).

(b) We will make use of the polarization formula

$$\int fg = \frac{1}{4}(\int (f+g)^2 + \int (f-g)^2)$$

Note that we can apply Parseval identity to $f+g$ and $f-g$ to obtain:

$$\int (f+g)^2 = 2\pi(a_0(f)+a_0(g))^2+\pi\sum\left[(a_n(f)+a_n(g))^2+(b_n(f)+b_n(g))^2\right]$$

$$\int (f-g)^2 = 2\pi(a_0(f)-a_0(g))^2+\pi\sum\left[(a_n(f)-a_n(g))^2+(b_n(f)-b_n(g))^2\right]$$

Adding the two formula together, and divide the result by 4, we get

$$\int fg = 2\pi a_0(f)a_0(g) + \pi\sum\left[a_n(f)a_n(g)+b_n(f)b_n(g)\right]$$

3. (a) We use the Gram-schmidt process. First, since $||1||_2 = \sqrt{\int_0^1 1^2} = 1$, we can take $\phi_1 = 1$. Similarly, we calculate: $\langle 1, x\rangle = \frac{1}{2}$, so we take

$$\phi_2 = \frac{x-\frac{1}{2}\cdot 1}{||x-\frac{1}{2}\cdot 1||_2} = \sqrt{3}(2x-1)$$

Next we calculate: $\langle 1, x^2\rangle = \frac{1}{3}, \langle\sqrt{3}(2x-1), x^2\rangle = \frac{\sqrt{3}}{6}$, so

$$\begin{aligned}\phi_3 =& \frac{x^2-\frac{1}{3}\cdot 1-\frac{\sqrt{3}}{6}\cdot\sqrt{3}(2x-1)}{||x^2-\frac{1}{3}\cdot 1-\frac{\sqrt{3}}{6}\cdot\sqrt{3}(2x-1)||_2}\\ =& \frac{x^2-x+\frac{1}{6}}{||x^2-x+\frac{1}{6}||_2}\\ =& \sqrt{5}(6x^2-6x+1)\end{aligned}$$

(b) The quadratic polynomial should be given as

$$\langle f,\phi_1\rangle\phi_1 + \langle f,\phi_2\rangle\phi_2 + \langle f,\phi_3\rangle\phi_3$$

Now, we have $\langle f,\phi_1\rangle = \log 2$, $\langle f,\phi_2\rangle = \sqrt{3}(2-3\log 2)$ and $\langle f,\phi_3\rangle = \sqrt{5}(13\log 2-9)$ so the required polynomial is

$$(\log 2)\cdot 1+\sqrt{3}(2-3\log 2)\cdot\sqrt{3}(2x-1)+\sqrt{5}(13\log 2-9)\cdot\sqrt{5}(6x^2-6x+1)$$

$$= 30(13\log 2-9)x^2 + 6(47-68\log 2)x + (75\log 2-51)$$

4. Consider the function f_1 defined in Homework 1 question 1a):

$$f(x) = \begin{cases} 0, & \text{if } x\in[0,\pi] \\ x, & \text{if } x\in(-\pi,0)\end{cases}$$

One can see the relevance of f_1 from the formula $|x| = -f(x) - f(-x)$. In homework 1, we calculate

$$f_1(x) \sim -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$$

so we also have

$$f_1(-x) \sim -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$$

Hence

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x$$

Now, we apply the Parseval Identity:

$$\begin{aligned}
\int_{-\pi}^{\pi} |x|^2 &= 2\pi\left(\frac{\pi}{2}\right)^2 + \pi\sum_{n=1}^{\infty}\left(\frac{4}{(2n-1)\pi}\right)^2 \\
\frac{2\pi^3}{3} &= \frac{\pi^3}{2} + \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \\
\frac{\pi^4}{96} &= \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \\
\frac{\pi^4}{96} &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}
\end{aligned}$$