MATH 3060 Tutorial 2

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1. Let f_2 be the 1-periodic extension of the function $\frac{x^2}{2} - \frac{x}{2} + \frac{1}{12}$ on $[0, 1]$. Last time we showed that

$$f(x) \sim \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(2\pi n x).$$

Show that we actually have

$$f(x) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(2\pi n x).$$

Is the convergence uniform?

2. (a) Show that

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{\sin((N+1/2)x)}{\sin(x/2)} dx = \lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{2\sin((N+1/2)x)}{x} dx$$

(b) Hence, show that

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$

3. Let $S \subset \mathbb{C}$ be the circle $\{x \in \mathbb{C} : |x| = 1\}$. Let $F : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function, and $f : S \to \mathbb{R}$ is the function such that $f(e^{ix}) = F(x)$ for all $x \in \mathbb{R}$. Suppose f is Riemann integrable on $[-\pi, \pi]$, show that

$$c_n(F) = \frac{1}{2\pi i} \int_S \frac{f(z)dz}{z^{n+1}}$$

We assume S is given the anti-clockwise orientation as usual.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function. Let N be a positive, we define

$$\hat{c}_n = \frac{1}{N} \sum_{k=1}^{N} f\left(\frac{2k\pi}{N}\right) e^{-\frac{2kn\pi i}{N}}$$

for $n = 1, 2, \ldots, N$

(a) Suppose f is Riemann integrable on $[0, 2\pi]$, show that

$$\lim_{N\to\infty} \hat{c}_n = c_n$$

(b) If $x = \frac{2r\pi}{N}$, $r \in \{1, 2, \ldots, N\}$, show that

$$f(x) = \sum_{n=1}^{N} \hat{c}_n e^{ix}$$