eg 3.3 $T : (0,1] \to (0,1]$ (Caution: $(0,1]$ is not complete)

$$x \mapsto \frac{x}{2}.$$

Clearly $|Tx - Ty| = \frac{1}{2}|x-y|$ $(\gamma = \frac{1}{2} < 1)$

$\therefore$ T is a contraction.

However, if $x \in (0,1]$ is a fixed point of T,

then $Tx = x \Leftrightarrow \frac{x}{2} = x \Leftrightarrow x = 0 \notin (0,1]$.

$\therefore$ T has no fixed point on $(0,1]$.

This example shows that "completeness" is necessary in the Contraction Mapping Principle.

eg 3.4: $S : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is complete)

$$x \mapsto x - \log(1+e^x).$$

Then $\frac{dS}{dx} = 1 - \frac{e^x}{1+e^x} = \frac{1}{1+e^x} < 1$ $(\& > 0)$

$\Rightarrow$ $|Sx - Sy| = \left|\frac{dS}{dx}(c)\right||x-y| < |x-y|$

(But there is no constant $\gamma < 1$ such that

$$|Sx - Sy| \le \gamma|x-y|, \ \forall x, y \in \mathbb{R} \quad (\text{Ex!})$$
)

Since $-\log(1+e^x) \neq 0 \quad \forall x \in \mathbb{R}$,

$$Sx \neq x \quad \forall x \in \mathbb{R}$$

i.e. no fixed point

This example shows that $\gamma < 1$ cannot replaced by $\gamma \leq 1$

#

eg3.5 Let $f : [0,1] \to [0,1]$ continuously differentiable with

$$|f'(x)| < 1 \text{ on } [0,1].$$

Then f has a fixed point in $[0,1]$.

Pf: By mean value theorem

$\forall x, y \in [0,1]$, $\exists z \in [0,1]$ s.t.

$$f(x) - f(y) = f'(z)(x-y)$$

$$\Rightarrow \quad |f(x) - f(y)| \leq |f'(z)|\,|x-y|$$

$$\leq \left(\sup_{[0,1]} |f'(z)|\right) |x-y|.$$

Since $|f'(z)| < 1$ & $f'(z)$ cts on $[0,1]$,

$$\gamma = \sup_{[0,1]} |f'(z)| \in [0,1).$$

If $r=0$, then $f \equiv c$ on $[0,1]$ with $c \in [0,1]$.

$(f:[0,1] \to [0,1])$

$\Rightarrow f(c)=c$.

If $r \neq 0$, then $r \in (0,1)$ &

$$|f(x)-f(y)| \leq r|x-y| \quad \forall x,y \in [0,1].$$

$\Rightarrow$ f is a contraction on the complete metric space $([0,1], \text{standard})$.

By contraction mapping principle,

f has a fixed point. #

Def: If a normed space $(X, \|\cdot\|)$ is complete as a metric space with respect to the induced metric

$$d(x,y) = \|x-y\|, \quad \forall x, y \in X.$$

Then it is called a Banach space.

eg. - $(\mathbb{R}^n, \|\cdot\|_p)$ $(p \geq 1)$ is a Banach space.

- $(C[a,b], \|\cdot\|_\infty)$ is a Banach space.

Thm 3.4 (Perturbation of Identity)

Let $(X, \|\cdot\|)$ be a Banach space, and

$$\Phi : \overline{B_r(x_0)} \to X \text{ satisfies } \Phi(x_0) = y_0$$

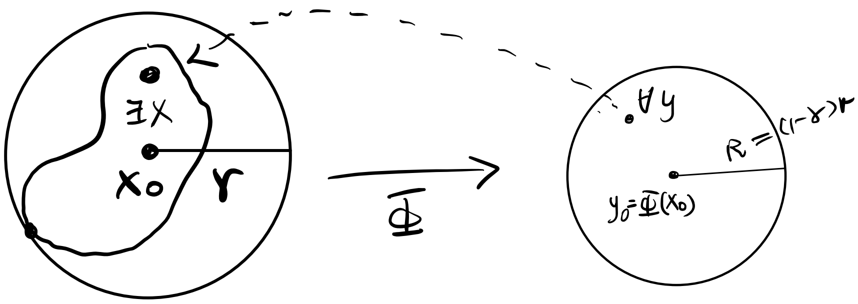
Suppose that $\Phi = Id_X + \Psi$ such that

$\exists$ constant $\gamma \in (0,1)$ such that

$$\|\Psi(x_2) - \Psi(x_1)\| \leq \gamma \|x_2 - x_1\|, \quad \forall x_1, x_2 \in \overline{B_r(x_0)}.$$

Then $\forall y \in \overline{B_R(y_0)}$, where $R = (1-\gamma) r$,

$\exists$ unique $x \in \overline{B_r(x_0)}$ such that $\Phi(x) = y$.



(i.e. Φ is Locally invertible)

Idea of proof :

$$y = \Phi(x) = (\mathrm{Id}_X + \Psi)(x) = x + \Psi(x)$$

$$\Longleftrightarrow \quad x = y - \Psi(x)$$

If $\forall y \in \overline{B_R(x_0)}$, define $Tx = y - \Psi(x)$. (note: different y gives different T)

Then $\quad y = \Phi(x) \Longleftrightarrow Tx = x \quad$ (i.e. x is a fixed point of T).

Proof: Define $\widetilde{\Phi} : \overline{B_r(0)} \to X$ by

$$\begin{aligned}
\widetilde{\Phi}(x) &= \Phi(x+x_0) - \Phi(x_0) \\
&= \big(x + x_0 + \Psi(x+x_0)\big) - \big(x_0 + \Psi(x_0)\big) \\
&= x + \left[\Psi(x+x_0) - \Psi(x_0)\right] \\
&= x + \widetilde{\Psi}(x)
\end{aligned}$$

Then $\widetilde{\Phi}(0) = 0$.

Further define, for any $y \in \overline{B_R(0)}$ $(R = (1-\gamma)r)$

the map
$$\begin{aligned} T : \overline{B_r(0)} &\to X \\ x &\mapsto y - \tilde{\Psi}(x) \end{aligned}$$

$$\left(\text{i.e.} \quad Tx = y - \tilde{\Psi}(x) : \overline{B_r(0)} \to X \right)$$

Then $\forall x \in \overline{B_r(0)}$,

$$\|Tx\| = \|y - \tilde{\Psi}(x)\| \leq \|y\| + \|\Psi(x+x_0) - \Psi(x_0)\|$$

$$\leq \|y\| + \gamma \|x\| \leq R + \gamma r = r \qquad (R = (1-\gamma)r)$$

$\therefore \quad T : \overline{B_r(0)} \to \overline{B_r(0)}$ (self map of $\overline{B_r(0)}$)

And $\forall x_1, x_2 \in \overline{B_r(0)}$,

$$\|Tx_1 - Tx_2\| = \|[y - (\Psi(x_1+x_0) - \Psi(x_0))] - [y - (\Psi(x_2+x_0) - \Psi(x_0))]\|$$

$$= \|\Psi(x_1+x_0) - \Psi(x_2+x_0)\|$$

$$\leq \gamma \|x_1 - x_2\|$$

Since $\gamma \in (0,1)$, $T : \overline{B_r(0)} \to \overline{B_r(0)}$ is a contraction.

Since $\overline{B_r(0)}$ is a closed subset and $(X, \|\cdot\|)$ is complete,

Prop 3.1 $\Rightarrow$ $\overline{B_r(0)}$ is also complete.

Hence one can apply Contraction Mapping Principle to conclude that $\exists$ unique $x \in \overline{B_r(0)}$ s.t.

$$Tx = x \quad \text{in } \overline{B_r(0)}.$$

i.e. $$x = y - \left(\Psi(x+x_0) - \Psi(x_0)\right)$$
$$= y - \left[\left(\Phi(x+x_0) - (x+x_0)\right) - \left(\Phi(x_0) - x_0\right)\right]$$
$$= y - \Phi(x+x_0) + \Phi(x_0) + x$$

$$\Leftrightarrow \quad \Phi(x+x_0) = y + y_0 \qquad (y_0 = \Phi(x_0))$$

Note that $y + y_0 \in \overline{B_R(y_0)}$ is arbitrary, and $x + x_0 \in \overline{B_r(x_0)}$, we've proved the Thm.

$\#$

<u>Remarks</u>

(1) Only need to assume Φ (and Ψ) defined on $B_r(x_0)$ (open ball) satisfying

$$\|\Psi(x_1) - \Psi(x_2)\| \le \gamma \|x_1 - x_2\|, \quad \gamma \in (0,1)$$

for $x_1, x_2 \in B_r(x_0)$ (open ball). Then it is easy to extend Φ (and Ψ) to $\overline{B_r(x_0)}$ and get the same inequality for all $x_1, x_2 \in \overline{B_r(x_0)}$. (Ex!)

(2) Actually one can prove more that

if $y \in B_R(y_0)$ (open ball), then the solution $x \in B_r(x_0)$ (open ball).

(check the details of the pf.)

(3) The Thm $\Rightarrow$ $\Phi^{-1}: \overline{B_R(y_0)} \to \overline{B_r(x_0)}$ exists.

<u>Claim</u> $\|\Phi^{-1}(y_1) - \Phi^{-1}(y_2)\| \leq \frac{1}{1-\gamma} \|y_1 - y_2\|, \ \forall y_1, y_2 \in \overline{B_R(y_0)}$.

In particular, Φ^{-1} is <u>uniformly continuous</u> (in fact "Lipschitz").

<u>Pf</u>: Let $x_i = \Phi^{-1}(y_i)$. Then x_i is the fixed point

such that $x_i = y_i - \Psi(x_i)$.

$$\Rightarrow \|\Phi^{-1}(y_1) - \Phi^{-1}(y_2)\| = \|(y_1 - \Psi(x_1)) - (y_2 - \Psi(x_2))\|$$

$$\leq \|y_1 - y_2\| + \|\Psi(x_1) - \Psi(x_2)\|$$

$$\leq \|y_1 - y_2\| + \gamma \|x_1 - x_2\|$$

$$= \|y_1 - y_2\| + \gamma \|\Phi^{-1}(y_1) - \Phi^{-1}(y_2)\|$$

$$\Rightarrow \|\Phi^{-1}(y_1) - \Phi^{-1}(y_2)\| \leq \frac{1}{1-\gamma} \|y_1 - y_2\| \quad \#$$

eg 3.6: $3x^4 - x^2 + x = \underbrace{-0.05}$ has a real root.

(<u>Observation</u>: (i) small and

(ii) $3x^4 - x^2 + x = 0$ has a root $x = 0$.

<u>Idea</u>: look for solution near $x=0$ using Thm 3.4.)

<u>Pf</u>: Let $\Phi(x) = x + (3x^4 - x^2) = x + \Psi(x)$

where $\Psi(x) = 3x^4 - x^2$ ("small" near $x=0$)

Then $\Phi(0) = 0$.

And for $x_1, x_2 \in \overline{B_r(0)}$ ($r > 0$ to be determined)

$$|\Psi(x_1) - \Psi(x_2)| = |(3x_1^4 - x_1^2) - (3x_2^4 - x_2^2)|$$

$$= |3(x_1^4 - x_2^4) - (x_1^2 - x_2^2)|$$

$$= |3(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3) - (x_1 + x_2)|\,|x_1 - x_2|$$

$$\leq (12r^3 + 2r)\,|x_2 - x_1|.$$

Hence, we need to choose $r > 0$ small enough such that

$$\gamma = 12r^3 + 2r < 1$$

Also, in order to include $-0.05 \in \overline{B_R(0)}$, we need

$$R = (1 - \gamma) r \geq 0.05.$$

A choice is $r = \frac{1}{4}$.

Then $\gamma = \frac{11}{16} < 1$ and $R = (1-\gamma)r = \frac{5}{64} \sim 0.078$.

By Thm 3.4, $\forall\, y \in \overline{B_{\frac{5}{64}}(0)}$, $\exists\, x \in \overline{B_{\frac{1}{4}}(0)}$ s.t. $\Phi(x) = y$

i.e. $x + 3x^4 - x^2 = y$.

In particular, $-0.05 \in \overline{B_{\frac{5}{64}}(0)}$, we has a root of

$$x + 3x^4 - x^2 = -0.05. \quad \#$$

One can generalize eg 3.6 to

<u>Prop 3.5</u>: Let $\Phi(x) = x + \Psi(x) : U \to \mathbb{R}^n$ be <u>C^1</u> on some open set $U \subset \mathbb{R}^n$ containing 0, such that

$$\Psi(0) = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{\partial \Psi_i}{\partial x_j}(x) = 0, \quad \forall\, i, j.$$

Then $\exists\, r > 0$ and $R > 0$ such that $\forall\, y \in B_R(0)$,

$\Phi(x) = y$ has a unique solution x in $B_r(0)$.

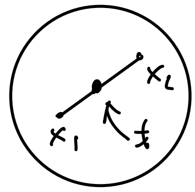
Pf: For $X_1, X_2 \in B_r(0)$ ($r>0$ to be determined)

($\leftarrow$ open ball using remark (1))

consider

$$\varphi_i(t) = \Psi_i(X_1 + t(X_2 - X_1)) \quad \text{for } t \in [0,1].$$

Then $\varphi_i(0) = \Psi_i(X_1)$, $\varphi_i(1) = \Psi_i(X_2)$.



$$\varphi_i'(t) = \frac{d}{dt}\Psi_i(X_1 + t(X_2 - X_1))$$

$$= \nabla \Psi_i(X_1 + t(X_2 - X_1)) \cdot (X_2 - X_1)$$

$$\Rightarrow |\Psi_i(X_2) - \Psi_i(X_1)| = |\varphi_i(1) - \varphi_i(0)| = \left|\int_0^1 \varphi_i'(t)\,dt\right|$$

$$\leq \int_0^1 |\nabla \Psi_i(X_1 + t(X_2 - X_1)) \cdot (X_2 - X_1)|\,dt$$

$$\leq \left(\int_0^1 |\nabla \Psi_i(X_1 + t(X_2 - X_1))|\,dt\right)|X_2 - X_1|$$

$$\leq |\nabla \Psi_i(X_1 + t^*(X_2 - X_1))|\,|X_2 - X_1|$$

(for some $t^* \in (0,1)$ by Mean Value Thm, since Ψ is C')

Note that $X_1, X_2 \in B_r(0) \Rightarrow X_1 + t^*(X_2 - X_1) \in B_r(0)$

Let

$$M_r = \sup_{X \in \overline{B_r(0)}} \left(\sum_{i,j=1}^{n} \left| \frac{\partial \Psi_i}{\partial X_j}(X) \right|^2 \right)^{1/2} > 0$$

(unless $\Psi \equiv 0$ in $\overline{B_r(0)}$ which is a trivial case.)

$$\Rightarrow \quad |\Psi(x_2)-\Psi(x_1)| = \sqrt{\sum_{i=1}^{n} |\Psi_i(x_2)-\Psi_i(x_1)|^2}$$

$$\leq \sqrt{\sum_{i=1}^{n} |\nabla\Psi_i(x_1+t^*(x_2-x_1))|^2}\ |x_2-x_1|$$

$$\leq \sqrt{n}\, M_r\, |x_2-x_1|$$

By $\lim_{x\to 0} \frac{\partial \Psi_i}{\partial x_j}(x) = 0$, $\forall\, i,j=1,\cdots,n$, and Ψ is C^1,

$\exists\ r>0$ s.t. $M_r \leq \frac{1}{2\sqrt{n}} \Rightarrow$

$$|\Psi(x_2)-\Psi(x_1)| \leq \frac{1}{2}|x_2-x_1|, \quad \forall\, x_1, x_2 \in B_r(0)$$

Take $R=(1-\frac{1}{2})r = \frac{r}{2}$.

By Thm 3.4 & Remarks (1) & (2),

$$\forall\, y\in B_{\frac{r}{2}}(0),\ \exists \text{ unique } x\in B_r(0) \text{ s.t. } \Phi(x)=y.$$

#

eg 3.7: (Integral Equation)

Let $K(x,t) \in C([0,1]\times[0,1])$ and

$$M = \|K\|_\infty = \max_{(x,t)\in[0,1]\times[0,1]} |K(x,t)| .$$

Then $\forall$ $g \in C[0,1]$ with $\|g\|_\infty < \frac{1}{8M}$,

$\exists$ unique solution $y \in C[0,1]$ with $\|y\|_\infty \le \frac{1}{4M}$ s.t.

$$\boxed{y(x) = g(x) + \int_0^1 K(x,t)\, y^2(t)\, dt}$$

Pf: Note that $(C[0,1], \|\cdot\|_\infty)$ is a Banach space.

Consider $\Phi : \overline{B^\infty_r(0)} \to C[0,1]$ ($r>0$ to be determined)

$$y \mapsto \Phi(y)$$

defined by $\Phi(y)(x) = y(x) - \int_0^1 K(x,t)\, y^2(t)\, dt, \quad \forall x \in [0,1]$

And let $\Psi(y) : \overline{B^\infty_r(0)} \to C[0,1]$ be defined by

$$\Psi(y)(x) = -\int_0^1 K(x,t)\, y^2(t)\, dt .$$

Then $\begin{cases} \Phi(y) = y + \Psi(y) & \text{and also} \\ \Phi(0) = 0 & (\text{\& } \Psi(0) = 0) \end{cases}$ (where 0 = zero function)

$\forall\, y_1, y_2 \in \overline{B_r^\infty(0)}$,

$$\|\Phi(y_1) - \Phi(y_2)\|_\infty = \max_{x\in[0,1]} \left| -\int_0^1 K(x,t)\, y_1^2(t)\,dt + \int_0^1 K(x,t)\, y_2^2(t)\,dt \right|$$

$$\leq \int_0^1 \left(\max_{x\in[0,1]} |K(x,t)|\right) \left|y_2^2(t) - y_1^2(t)\right| dt$$

$$\leq M\, \|y_2^2 - y_1^2\|_\infty$$

$$\leq M\, \|y_2 + y_1\|_\infty \|y_2 - y_1\|_\infty$$

$$\leq 2rM\, \|y_2 - y_1\|_\infty$$

Choose $r = \frac{1}{4M}$, then

$$\|\Phi(y_1) - \Phi(y_2)\|_\infty \leq \frac{1}{2} \|y_1 - y_2\|_\infty , \qquad \forall\, y_1, y_2 \in \overline{B^\infty_{\frac{1}{4M}}(0)} .$$

Hence Thm 3.4 $\Rightarrow$

$\forall\, g \in \overline{B_R^\infty(0)}$ with $R = (1-\frac{1}{2})r = \frac{1}{2}\cdot\frac{1}{4M} = \frac{1}{8M} > 0$,

$\exists!\, y \in \overline{B^\infty_{\frac{1}{4M}}(0)}$ s.t. $\Phi(y) = g$

i.e. $y(x) - \int_0^1 K(x,t)\, y^2(t)\,dt = g(x), \ \forall\, x\in[0,1]$

which is the required solution to the integral equation. $\divideontimes$