MATH3060 HW4 Due date: Oct 19, 2022 (at 11:00 am)

1. Recall that $l_2 = \{ x = (x_1, x_2, \cdots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty, \ x_i \in \mathbb{R} \}$ has a metric $d_2(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2}$.

Show that the set

$$H = \{ x = (x_1, x_2, \cdots) : |x_i| \leq \frac{1}{i}, \ \forall i = 1, \cdots, \infty \}$$

is a closed subset in (l_2, d_2).

2. Let (X, d) be a metric space.

(a) If A is a closed subset of X and $x_0 \in X \setminus A$. Show that there is a continuous function f on X such that $f(x_0) = 0$ and $f(x) = 1, \ \forall x \in A$.

(b) If A, B are disjoint closed subsets of X, show that there exists a continuous function g on X such that $g(x) = 0, \ \forall x \in A$ and $g(x) = 1, \ \forall x \in B$.

3. Identify the boundary, interior, and closure of the following sets in the indicated metric space:

(a) $\bigcup_{k=1}^{\infty} \left(\frac{1}{k+1}, \frac{1}{k}\right)$ in ($\mathbb{R}$, standard metric)

(b) $\mathbb{R}^2 \setminus \{(\frac{1}{n}, 0) : n = 1, 2, 3, \cdots\}$ in ($\mathbb{R}^2$, Euclidean metric)

(c) $\{f \in C[0,1] : f(0) = f(1)\}$ in $(C[0,1], d_\infty)$

4. Prove the generalized Hölder's inequality:

$\forall f_i \in R[a,b]$, $i = 1, \cdots, n$,

$$\int_a^b |f_1 f_2 \cdots f_n| \, dx \le \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_n\|_{p_n}$$

where $p_i > 1$, for all $i = 1, \cdots, n$, and satisfy

$$\sum_{i=1}^{n} \frac{1}{p_i} = 1.$$

(End)