MATH3060 HW3 Due date: Oct 14, 2022 (at 11:00 am)

1. Show that $d: X \times X \to \mathbb{R}$ $(X \neq \phi)$ is a metric on X if and only if d satisfies the following 2 conditions:

   (i) $d(x,y) \geq 0$, $\forall x, y \in X$ & "equality holds $\Leftrightarrow x = y$"

   (ii) $d(x,y) \leq d(z,x) + d(z,y)$ $\quad \forall x, y, z \in X$

2. (a) Let $l_1 = \{ x = (x_1, x_2, \cdots) : \sum_{i=1}^{\infty} |x_i| < \infty,\ x_i \in \mathbb{R} \}$.

   Show that $d_1(x,y) = \sum_{i=1}^{\infty} |x_i - y_i|$ is a metric on l_1

   (b) Let $l_2 = \{ x = (x_1, x_2, \cdots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty,\ x_i \in \mathbb{R} \}$.

   Show that $d_2(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{1/2}$ is a metric on l_2

   (c) Let $l_\infty = \{ x = (x_1, x_2, \cdots) : \sup_i |x_i| < \infty,\ x_i \in \mathbb{R} \}$.

   Show that $d_\infty(x,y) = \sup_i |x_i - y_i|$ is a metric on l_∞

   (d) Show that as sets,

   $$l_1 \subset l_2 \subset l_\infty .$$

3. Determine whether the following mappings between metric spaces are continuous:

($\mathbb{R}$ always equipped with the standard metric $d(x,y) = |x-y|$, d_1, d_∞ on $C[a,b]$ as in the lecture notes.)

(a) $\Phi : (C[a,b], d_1) \to \mathbb{R}$ given by

$$\Phi(f) = \int_a^b \sqrt{1+f^2(x)}\, dx$$

(b) $\Phi : (C[a,b], d_\infty) \to \mathbb{R}$ with same Φ as in (a).

(c) $\Psi : (C[-1,1], d_1) \to \mathbb{R}$ given by

$$\Psi(f) = f(0).$$

(d) $\Psi : (C[-1,1], d_\infty) \to \mathbb{R}$ with same Ψ as in (c).

4. Show that for any $\alpha \in \mathbb{R}$, the set

$$\{ f \in C[a,b] : f(x) \geq \alpha,\ \forall\, x \in [a,b] \}$$

is closed in $(C[a,b], d_\infty)$.

(End)