MATH3060 HW2 Due date: Oct 5, 2022 (at 11:00 am)

1. Let f be a C^∞ 2π-periodic function on $[-\pi,\pi]$.
   Show that the Fourier coefficients

$$|a_n| = o\left(\frac{1}{n^k}\right) \quad \text{and} \quad |b_n| = o\left(\frac{1}{n^k}\right)$$

as $n \to \pm\infty$ for every k.

2. (a) Let f, g and $h \in R[a,b]$. Show that

$$\|f-g\|_2 \leq \|f-h\|_2 + \|h-g\|_2 .$$

When does the equality sign hold?

(b) Let f, g be 2π-periodic functions integrable on $[-\pi,\pi]$. Show that

$$\int_{-\pi}^{\pi} fg = 2\pi\, a_0(f)\, a_0(g) + \pi \sum_{n=1}^{\infty} \left[a_n(f)\, a_n(g) + b_n(f)\, b_n(g)\right]$$

where a_0, a_n, b_n are corresponding Fourier coefficients.

(cont. on next page)

3 (No need to use Fourier Series)

(a) Let $\mathcal{S} = \text{Span}\{1, x, x^2\}$ in $C[0,1]$. Find an orthonormal set, in L^2-sense, $\{\varphi_1, \varphi_2, \varphi_3\}$ in $\mathcal{S}$

i.e. $\langle \varphi_i, \varphi_j \rangle_2 = \delta_{ij} \quad \forall i, j = 1, 2, 3$

such that $\deg \varphi_j \leq j-1$, $j = 1, 2, 3$

(Note: elements in $\mathcal{S}$ are polynomals of degree ≤ 2)

(b) Find the quadratic polynomial that minimizing the L^2-distance from the function $f(x) = \frac{1}{1+x} \in C[0,1]$ to $\mathcal{S}$.

4. Using the Parseval Identity and the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$ to show that

$$\frac{\pi^4}{96} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}.$$

(End)