THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2068 Mathematical Analysis II (Spring 2023)
Suggested Solution of Homework 7

Q2

When $x = 0$, it is trivial.
When $x \neq 0$, $\lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{x/n}{1/n^2+x^2} = \frac{0}{x^2} = 0$.

Q12

Fix $\epsilon > 0$. By AP, there exists $N \in \mathbb{N}$ such that for any $n > N$, $\frac{1}{na} < \epsilon$. For any $n > N$, for $x \in [a, \infty]$, $0 < \frac{nx}{1+n^2x^2} < \frac{nx}{n^2x^2} = \frac{1}{nx} \leq \frac{1}{na} < \epsilon$. Hence, $\frac{nx}{1+n^2x^2}$ converges uniformly to 0 on $[a, \infty]$.
Take $x_n = \frac{1}{n}$. Then $\frac{nx_n}{1+n^2x_n^2} = \frac{1}{2}$ for any $n \in \mathbb{N}$. Hence, $\frac{nx}{1+n^2x^2}$ does not converge uniformly to 0 on $[0, \infty]$.

Q22

Fix $\epsilon > 0$. By AP, there exists $N \in \mathbb{N}$ such that for any $n > N$, $\frac{1}{n} < \epsilon$. For any $n > N$, for $x \in \mathbb{R}$, $|f_n(x) - f(x)| = \frac{1}{n} < \epsilon$. Hence, f_n converges uniformly to f on $\mathbb{R}$.
Note that $f_n^2(x) = x^2 + \frac{2x}{n} + \frac{1}{n^2}$. Clearly, f_n^2 converges pointwisely to x^2. By the uniqueness of the limit, it suffices to show f_n does not converge uniformly to x^2 on $\mathbb{R}$. Take $x_n = n$. Then $|f_n^2(x_n) - x_n^2| = 2 + \frac{1}{n^2} > 2$ for any $n \in \mathbb{N}$. Hence, f_n does not converge uniformly to f on $\mathbb{R}$.