THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2068 Mathematical Analysis II (Spring 2023)
Suggested Solution of Homework 6

Q18

Let $||f||_\infty := \sup\{f(x) : x \in [a,b]\}$. Since $f \geq 0$, if $||f||_\infty = 0$, then $f \equiv 0$. The equality holds trivially.
Suppose $||f||_\infty > 0$. Dividing by $||f||_\infty > 0$ on both sides and replacing f by $\frac{f}{||f||_\infty}$, without loss of generality, we can assume $0 \leq f \leq 1$ and $||f||_\infty = 1$.
On one hand, since $f^n \leq 1$, then $M_n \leq (b-a)^{1/n}$, which implies $\limsup_{n\to\infty} M_n \leq 1$.
On the other hand, for any $\epsilon > 0$, since f is continuous, there exists a subinterval $[c,d] \subset [a,b]$ such that $f \geq 1-\epsilon$ on $[c,d]$. Since $f^n \geq 0$ on $[a,b]$ and $f^n \geq (1-\epsilon)^n$ on $[c,d]$, then $M_n \geq (1-\epsilon)(d-c)^{1/n}$, which implies $\liminf_{n\to\infty} M_n \geq 1-\epsilon$. Since ϵ is arbitrary, $\liminf_{n\to\infty} M_n \geq 1$.
Hence, $\lim_{n\to\infty} M_n = 1 = ||f||_\infty$.

Q21

(a) It follows from the fact $(tf \pm g) \geq 0$ and the monotonicity of integrals.

(b) Fix $t > 0$. By (a),
$$0 \leq \int_a^b (tf \pm g) = t^2 \int_a^b f^2 \pm 2t \int_a^b fg + \int_a^b g^2.$$
Since $t > 0$,
$$\pm 2 \int_a^b fg \leq t \int_a^b f^2 + \frac{1}{t} \int_a^b g^2,$$
i.e.,
$$2\left|\int_a^b fg\right| \leq t \int_a^b f^2 + \frac{1}{t} \int_a^b g^2.$$

(c) Suppose $\int_a^b f^2 = 0$. By (b),
$$2\left|\int_a^b fg\right| \leq \frac{1}{t} \int_a^b g^2$$
for any $t > 0$. Letting $t \to \infty$ gives us $\left|\int_a^b fg\right| = 0$, which implies $\int_a^b fg = 0$.

(d) By triangle inequality, $\left|\int_a^b fg\right| \leq \int_a^b |fg|$, which implies the first inequality.
If $\int_a^b f^2 = 0$, replacing f and g by $|f|$ and $|g|$ in (c) gives $\int_a^b |fg| = 0$. The second inequality holds trivially. The same argument works in the case $\int_a^b g^2 = 0$.
Suppose $\int_a^b f^2 > 0$ and $\int_a^b g^2 > 0$. Replace f and g by $|f|$ and $|g|$ in (b), we attain
$$2\int_a^b |fg| \leq t \int_a^b f^2 + \frac{1}{t} \int_a^b g^2$$
for any $t > 0$. In particular, taking $t = \left(\frac{\int_a^b g^2}{\int_a^b f^2}\right)^{1/2}$ gives us
$$2\int_a^b |fg| \leq 2\left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}.$$