THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2068 Mathematical Analysis II (Spring 2023)
Suggested Solution of Homework 1

Q4

Note that $\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$. When x is rational, $\frac{f(x)}{x} = \frac{x^2}{x} = x$; when x is irrational, $\frac{f(x)}{x} = \frac{0}{x} = 0$. In both cases, $|\frac{f(x)-f(0)}{x-0}| \leq |x|$.
For any $\epsilon > 0$, take $\delta = \epsilon$. For any $x \in \mathbb{R}$ with $|x| < \delta = \epsilon$, we have $|\frac{f(x)-f(0)}{x-0}| \leq |x| < \epsilon$.
Therefore, $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$. Hence, $f'(0) = 0$.

Q7

If $f'(c) = 0$, i.e., $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in (c-\delta, c+\delta)$, $|\frac{g(x)-g(c)}{x-c}| = |\frac{|f(x)|-|f(c)|}{x-c}| = \frac{|f(x)|}{|x-c|} = |\frac{f(x)-f(c)}{x-c}| < \epsilon$. Therefore, $g'(c) = 0$.
If $f'(c) = L \neq 0$, i.e., $\lim_{x\to c} \frac{f(x)}{x-c} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = L$. Since $||\frac{f(x)}{x-c}| - |L|| \leq |\frac{f(x)}{x-c} - L|$, $\lim_{x\to c} |\frac{f(x)}{x-c}| = |L|$. We wish to apply Sequential Criterion to show $\lim_{x\to c} \frac{g(x)-g(c)}{x-c} = \lim_{x\to c} \frac{|f(x)|}{x-c}$ does not exist. Take $x_n = c + \frac{1}{n}$. Since $x_n \to c$ and $x_n > c$, then $\frac{|f(x_n)|}{x_n - c} = |\frac{f(x_n)}{x_n - c}| = |L|$ as $n \to \infty$. Take $y_n = c - \frac{1}{n}$. Since $y_n \to c$ and $y_n < c$, then $\frac{|f(y_n)|}{y_n - c} = -|\frac{f(y_n)}{y_n - c}| = -|L|$. If $M := \lim_{x\to c} \frac{|f(x)|}{x-c}$ exists, then $|L| = M = -|L|$. But $L \neq 0$ implies $|L| \neq -|L|$. We arrived at a contradiction!

Q17

By definition, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $x \in (c-\delta, c+\delta)$, $|\frac{f(x)-f(c)}{x-c} - f'(c)| < \epsilon$, i.e., $|f(x) - f(c) - (x-c)f'(c)| < \epsilon|x-c|$. In particular, for $u, v \in I$ satisfying $c - \delta - u \leq c \leq v < c + \delta$, $|f(u) - f(c) - (u-c)f'(c)| < \epsilon(c-u)$ and $|f(v) - f(c) - (v-c)f'(c)| < \epsilon(v-c)$. By Triangle Inequality, $|f(v) - f(u) - (v-u)f'(c)| \leq |f(u) - f(c) - (u-c)f'(c)| + |f(v) - f(c) - (v-c)f'(c)| < \epsilon(c-u) + \epsilon(v-c) = (v-u)\epsilon$.