Mar. 2.

10. Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and let $v: [c,d] \to \mathbb{R}$ be differentiable on $[c,d]$ with $v([c,d]) \subset [a,b]$.
If we define $G(x) := \int_a^{v(x)} f$, show that $G'(x) = f(v(x)) \cdot v'(x)$ for all $x \in [c,d]$.

Proof. $$\lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_{v(x)}^{v(x+\Delta x)} f}{\Delta x}$$

Mean Value Theorem for Integrals

$$\lim_{\Delta x \to 0} \frac{f(c)\left(v(x+\Delta x) - v(x)\right)}{\Delta x}, \quad c \in [v(x), v(x+\Delta x)]$$

$$= f(v(x)) \cdot v'(x).$$

∎

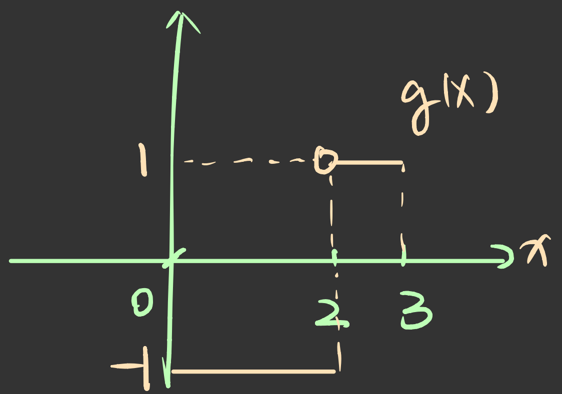
13. $g(x) = \begin{cases} -1 & 0 \le x < 2 \\ 1 & 2 \le x \le 3 \end{cases}$

Find $G(x) = \int_0^x g$, for $0 \le x \le 3$

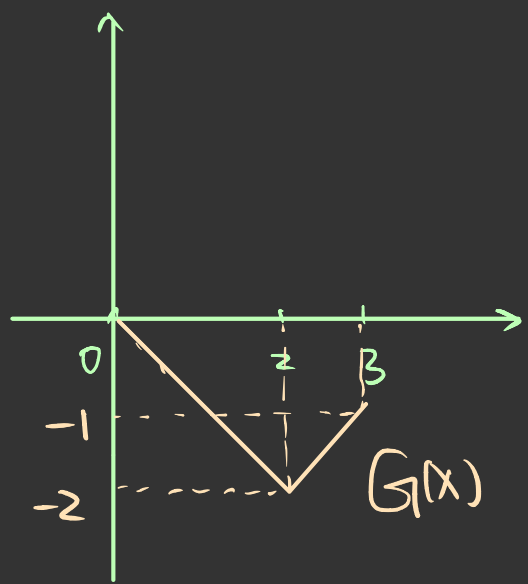
Sketch g and G.

Does $G'(x) = g(x)$ on $x \in [0, 3]$.

Sol.



$G(x) = \begin{cases} -x, & 0 \le x \le 2 \\ x - 4, & 2 \le x \le 3 \end{cases}$



$G'(2)$ doesn't exist.

∎

17. Use the following argument to prove

Theorem 7.3.8. ($\int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$

$f: I \to \mathbb{R}$ continuous,

$\varphi \in C^1[\alpha,\beta]$, $\varphi(\overset{[\alpha,\beta]}{J}) \subset I$)

Define $F(u) := \int_{\varphi(\alpha)}^{u} f(x)\,dx$ for $u \in I$, and

$H(t) := F(\varphi(t))$ for $t \in J$. Show that

$H'(t) = f(\varphi(t))\varphi'(t)$ for $t \in J$ and

$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = F(\varphi(\beta)) = H(\beta) = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt.$

Proof. $H'(t) = \lim_{\Delta t \to 0} \dfrac{\int_{\varphi(t)}^{\varphi(t+\Delta t)} f\,dx}{\Delta t} \underset{\substack{\uparrow \\ 7.3.10.}}{=} f(\varphi(t))\varphi'(t)$

$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = F(\varphi(\beta)) = H(\beta) = \int_\alpha^\beta H'(t)\,dt = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt.$

(↑ Fundamental Thm of Calculus) ▥