MATH 2060 TUT08

9. Let f_1 and f_2 be bounded functions on $[a, b]$. Show that $L(f_1) + L(f_2) \le L(f_1 + f_2)$.

Pf: Let $\varepsilon > 0$.

By def. of lower integral, ($L(f_1), L(f_2) \in \mathbb{R}$ since f_1, f_2 are bounded)

$\exists P_1 \in \mathcal{P}([a,b])$ s.t. $L(f_1, P_1) > L(f_1) - \varepsilon$

$\exists P_2 \in \mathcal{P}([a,b])$ s.t. $L(f_2, P_2) > L(f_2) - \varepsilon$

Let $P_1 \cup P_2$ be the partition obtained by combining the pts of P_1 and P_2.

Then $P_1 \cup P_2$ is a refinement of both P_1, P_2.

By Lemma 7.4.2, we have

$$L(f_1; P_1) \le L(f_1; P_1 \cup P_2)$$
$$L(f_2; P_2) \le L(f_2; P_1 \cup P_2)$$

So
$$\begin{aligned} L(f_1) + L(f_2) - 2\varepsilon &< L(f_1; P_1) + L(f_2; P_2) \\ &\le L(f_1; P_1 \cup P_2) + L(f_2; P_1 \cup P_2) \\ &\le L(f_1 + f_2; P_1 \cup P_2) \quad ? \end{aligned}$$

Claim: $L(f_1; Q) + L(f_2; Q) \le L(f_1 + f_2; Q) \quad \forall Q \in \mathcal{P}([a,b])$.

Pf: Suppose $Q = (x_0, x_1, \dots, x_n)$.

Then, by §2.4 Ex 8,

$$\inf_{[x_{k-1}, x_k]} f_1 + \inf_{[x_{k-1}, x_k]} f_2 \le \inf_{[x_{k-1}, x_k]} (f_1 + f_2) \qquad \forall k = 1, \dots, n.$$

$$\Rightarrow \sum_{k=1}^{n} \inf_{[x_{k-1}, x_k]} f_1 (x_k - x_{k-1}) + \sum_{k=1}^{n} \inf_{[x_{k-1}, x_k]} f_2 (x_k - x_{k-1}) \le \sum_{k=1}^{n} \inf_{[x_{k-1}, x_k]} (f_1 + f_2)(x_k - x_{k-1})$$

i.e. $L(f_1; Q) + L(f_2; Q) \le L(f_1 + f_2; Q)$ //

Now, $L(f_1) + L(f_2) - 2\varepsilon \le L(f_1 + f_2; P_1 \cup P_2) \le L(f_1 + f_2)$.

The result follows since $\varepsilon > 0$ is arbitrary. //

12. Let $f(x) = x^2$ for $0 \le x \le 1$. For the partition $\mathcal{P}_n := (0,\ 1/n,\ 2/n, \ldots, (n-1)/n, 1)$, calculate $L(f, \mathcal{P}_n)$ and $U(f, \mathcal{P}_n)$, and show that $L(f) = U(f) = \frac{1}{3}$. (Use the formula $1^2 + 2^2 + \cdots + m^2 = \frac{1}{6}m(m+1)(2m+1)$.)

Ans: Since f is an increasing fcn,
its sup and inf on $[\frac{k-1}{n}, \frac{k}{n}]$ are attained at
the right and left end pts, respectively

$$\Rightarrow \quad m_k = \inf_{[\frac{k-1}{n}, \frac{k}{n}]} f = \left(\frac{k-1}{n}\right)^2, \qquad M_k = \sup_{[\frac{k-1}{n}, \frac{k}{n}]} f = \left(\frac{k}{n}\right)^2$$

Moreover, since $x_k - x_{k-1} = \frac{1}{n}$ for $k = 1, \ldots, n$,
we have

$$L(f, P_n) = \frac{1}{n^3}\left[0^2 + 1^2 + \cdots + (n-1)^2\right] = \frac{1}{n^3} \cdot \frac{1}{6}(n-1)(n)(2n-1)$$
$$= \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$
$$U(f, P_n) = \frac{1}{n^3}\left[1^2 + 2^2 + \cdots + n^2\right] = \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1)$$
$$= \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

So $L(f) \geq \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) \quad \forall n \in \mathbb{N} \quad \Rightarrow \quad L(f) \geq \frac{2}{6} = \frac{1}{3}$

$U(f) \leq \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \quad \forall n \in \mathbb{N} \quad \Rightarrow \quad U(f) \leq \frac{1}{3}$

Since $\frac{1}{3} \leq L(f) \leq U(f) \leq \frac{1}{3}$,
we conclude that

$$L(f) = U(f) = \frac{1}{3}$$

//

21. Show that if (f_n), (g_n) converge uniformly on the set A to f, g, respectively, then $(f_n + g_n)$ converges uniformly on A to $f + g$.

Pf: Let $\varepsilon > 0$ be given.

Since $f_n \rightrightarrows f$ on A,

$\exists\, N_1 \in \mathbb{N}$ s.t. if $n \geq N_1$, then

$$|f_n(x) - f(x)| < \varepsilon/2 \qquad \forall x \in A.$$

Since $g_n \rightrightarrows g$ on A,

$\exists\, N_2 \in \mathbb{N}$ s.t. if $n \geq N_2$, then

$$|g_n(x) - g(x)| < \varepsilon/2 \qquad \forall x \in A$$

Take $N := \max\{N_1, N_2\}$.

Now, if $n \geq N$, then, $\forall x \in A$,

$$\begin{aligned}
|(f_n + g_n)(x) - (f+g)(x)| &= |(f_n(x) + g_n(x)) - (f(x) + g(x))| \\
&= |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\
&\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

Hence $(f_n + g_n) \rightrightarrows (f+g)$ on A //

Problem 11.6 Determine whether the sequence $\{f_n\}$ converges uniformly on D.

a) $f_n(x) = \dfrac{1}{1+(nx-1)^2}$ $\quad D = [0,1]$

b) $f_n(x) = nx^n(1-x)$ $\quad D = [0,1]$

c) $f_n(x) = \arctan\left(\dfrac{2x}{x^2+n^3}\right)$ $\quad D = \mathbb{R}$

Lemma 8.1.5 (f_n) does not converge uniformly on A to f iff $\exists\, \varepsilon_0 > 0$, $\exists$ subseq (f_{n_k}) of (f_n), $\exists\, (x_k) \in A$ s.t

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 \quad \forall\, k \in \mathbb{N}.$$

a) Note $\displaystyle\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+(nx-1)^2} = \left\{ \begin{array}{ll} 0 & x \in (0,1] \\ \frac{1}{2} & x = 0. \end{array} \right\} =: f(x)$

(We can actually conclude that the convergence is not uniform, for otherwise the limit fun is also cts)

Take $\varepsilon_0 = 1$ and $x_n = \frac{1}{n}$ $\forall\, n \in \mathbb{N}$.

Then $|f_n(x_n) - f(x_n)| = |f_n(x_n) - 0| = 1 \geq \varepsilon_0 \quad \forall\, n \in \mathbb{N}$.

Hence (f_n) does not converge uniformly on $[0,1]$ //

b) Note $\displaystyle\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx^n(1-x) = 0 \quad \forall\, x \in [0,1]$.

(How can we find (x_n) in this case?)

Note $\forall\, n \geq 2$,

$$f_n'(x) = n^2 x^{n-1}(1-x) + nx^n(-1) = nx^{n-1}(n-(n+1)x)$$

$\Rightarrow f_n'(x) = 0 \iff x = 0$ or $x = \frac{n}{n+1}$.

It is easy to check that

$$\sup\{f_n(x) : x \in [0,1]\} = f_n\left(\tfrac{n}{n+1}\right).$$

Now $\displaystyle\lim_{n\to\infty} f_n\left(\tfrac{n}{n+1}\right) = \lim_{n\to\infty} \left(\tfrac{n}{n+1}\right)^{n+1} = \lim_{n\to\infty} \left(1 - \tfrac{1}{n+1}\right)^{n+1} = \frac{1}{e} \neq 0$

So $\displaystyle\lim_{n\to\infty} \left|f_n\left(\tfrac{n}{n+1}\right) - f\left(\tfrac{n}{n+1}\right)\right| = \frac{1}{e} \neq 0$

Therefore (f_n) does not converge uniformly on $[0,1]$

c) Note $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \arctan\left(\frac{2x}{x^2+n^3}\right) = 0$

As in b), $f_n'(x) = \dfrac{1}{\left(\frac{2x}{x^2+n^3}\right)^2+1} \cdot \dfrac{2(x^2+n^3)-2x(2x)}{(x^2+n^3)^2}$

$$= \frac{2n^3-2x^2}{(x^2+n^3)^2+4x^2}$$

$\Rightarrow f_n'(x) = 0 \iff x = \pm n\sqrt{n}$

Since $\lim_{x\to\pm\infty} |f_n(x)| = 0$, it is easy to check that
$\sup\{|f_n(x) - 0| : x \in \mathbb{R}\} = |f_n(\pm n\sqrt{n})| = \arctan\left(\frac{1}{n\sqrt{n}}\right)$

Thus $\|f_n - 0\|_{\mathbb{R}} = \arctan\left(\frac{1}{n\sqrt{n}}\right) \longrightarrow 0$ as $n \to \infty$

Hence (f_n) converges uniformly on $\mathbb{R}$ to $f := 0$ //