THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2060 Mathematical Analysis II (Spring 2023)
Suggested Solution of Homework 3

Section 6.4

4. Show that if $x > 0$, then $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$.

   Solution. Let $f(x) = \sqrt{1+x}$. Then, for any $x > -1$,

$$f'(x) = \frac{1}{2\sqrt{1+x}}, \quad f''(x) = -\frac{1}{4(1+x)^{3/2}}, \quad f'''(x) = \frac{3}{8(1+x)^{5/2}}.$$

   Fix $x > 0$. By Taylor's Theorem, there exists $c_1 \in (0, x)$ such that

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(c_1)}{2!}(x-0)^2 \\ &= 1 + \frac{1}{2}x - \frac{1}{8(1+c_1)^{3/2}}x^2. \end{aligned}$$

   Since $-\frac{1}{8(1+c_1)^{3/2}}x^2 < 0$, we have $\sqrt{1+x} \leq 1 + \frac{1}{2}x$.
   Similarly, there exists $c_2 \in (0, x)$ such that

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(c_2)}{3!}(x-0)^3 \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16(1+c_2)^{5/2}}x^3. \end{aligned}$$

   Since $\frac{1}{16(1+c_2)^{5/2}}x^3 > 0$, we have $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x}$. □

9. If $g(x) := \sin x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$ for each fixed x_0 and x.

   Solution. For fixed x_0 and x, the n-th remainder term in Taylor's Theorem is

$$R_n(x) = \frac{g^{(n+1)}(c_n)}{(n+1)!}(x - x_0)^{n+1} \quad \text{for some } c_n \text{ between } x_0 \text{ and } x.$$

   Since $g^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, we have $|g^{(n+1)}(c_n)| \leq 1$ and hence

$$|R_n(x)| \leq \frac{|x - x_0|^{n+1}}{(n+1)!} =: a_n.$$

   Since $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{|x-x_0|}{n+1} = 0 < 1$, the ratio test yields $\lim\limits_{n\to\infty} a_n = 0$.
   Therefore, $\lim\limits_{n\to\infty} R_n(x) = 0$ by the squeeze theorem. □

10. Let $h(x) := e^{-1/x^2}$ for $x \neq 0$ and $h(0) := 0$. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that the remainder term in Taylor's Theorem for $x_0 = 0$ does not converge to zero as $n \to \infty$ for $x \neq 0$.

 Solution. First, we show that $\lim_{x\to 0} h(x)/x^k = 0$ for any $k \in \mathbb{N}$. By successive application of L'Hospital's Rule,

$$\lim_{y\to+\infty} \frac{y^k}{e^y} = \lim_{y\to+\infty} \frac{ky^{k-1}}{e^y} = \cdots = \lim_{y\to+\infty} \frac{k!}{e^y} = 0 \quad \text{for any } k \in \mathbb{N}.$$

 Let $y = 1/x^2$. Then $y \to +\infty$ as $x \to 0$. Hence, for any $k \in \mathbb{N}$,

$$\lim_{x\to 0} \frac{h(x)}{x^k} = \lim_{x\to 0} \frac{(1/x^2)^k}{e^{1/x^2}} \cdot x^k = 0.$$

 Next, we calculate $h^{(n)}(x)$ for $x \neq 0$. Clearly $h(x) = e^{-1/x^2}$ is infinitely differentiable for $x \neq 0$. By applying Leibniz's rule to $h'(x) = \frac{2}{x^3}e^{-1/x^2} = \frac{2}{x^3}h(x)$, we have

$$h^{(n+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2}{x^3}\right)^{(n-k)} h^{(k)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{(n-k+2)!}{x^{n-k+3}} h^{(k)}(x) \quad (*)$$

 for any $x \neq 0$ and integer $n \geq 0$.

 Now, we prove by induction on n that

 (i) $\lim_{x\to 0} \dfrac{h^{(n)}(x)}{x^m}$ for any $m \in \mathbb{N}$;

 (ii) $h^{(n)}(0) = 0$.

 The case $n = 0$ is obviously true. Suppose (i) and (ii) are true for n. Then $(*)$ gives

$$\lim_{x\to 0} \frac{h^{(n+1)}(x)}{x^m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}(n-k+2)! \left(\lim_{x\to 0} \frac{h^{(k)}(x)}{x^{n-k+3+m}} \right) = 0.$$

 Moreover,

$$h^{(n+1)}(0) = \lim_{x\to 0} \frac{h^{(n)}(x) - h^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{h^{(n)}(x)}{x} = 0.$$

 This completes the induction.

 Finally, the remainder term in Taylor's Theorem is given by

$$R_n(x) = h(x) - \sum_{k=0}^{n} \frac{h^{(k)}(0)}{k!} x^k = h(x),$$

 and so $\lim_{x\to 0} R_{n+1}(x) = h(x) \neq 0$ for $x \neq 0$. □

11. If $x \in [0, 1]$ and $n \in \mathbb{N}$, show that

$$\left| \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} \right) \right| < \frac{x^{n+1}}{n+1}.$$

 Use this to approximate $\ln 1.5$ with an error less than 0.01. Less than 0.001.

Solution. Let $f(x) = \ln(1+x)$. Then f is infinitely differentiable on $(-1,\infty)$ and

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \quad \text{for } x > -1, n \in \mathbb{N}.$$

Fix $x \in (0,1]$ and $n \in \mathbb{N}$. By Taylor's Theorem, $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n},$$

and for some $c_n \in (0,x)$,

$$R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}x^{n+1} = \frac{1}{n+1} \cdot \frac{(-1)^n}{(1+c_n)^{n+1}}x^{n+1}.$$

The inequality follows since $|R_n(x)| < \dfrac{x^{n+1}}{n+1}$.

(**Remark**: The inequality is not true when $x = 0$.)

Put $x = 0.5$, we have

$$|\ln 1.5 - P_n(0.5)| < \frac{(0.5)^{n+1}}{n+1}.$$

When $n = 4$, $\frac{(0.5)^{n+1}}{n+1} = 0.0625 < 0.01$. So, with an error less than 0.01,

$$\ln 1.5 \approx P_4(0.5) \approx 0.4010416667.$$

When $n = 7$, $\frac{(0.5)^{n+1}}{n+1} \approx 0.0004882 < 0.001$. So, with an error less than 0.001,

$$\ln 1.5 \approx P_7(0.5) \approx 0.4058035714.$$

□

15. Let f be continuous on $[a,b]$ and assume the second derivative f'' exists on (a,b). Suppose that the graph of f and the line segment joining the points $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0)$ where $a < x_0 < b$. Show that there exists a point $c \in (a,b)$ such that $f''(c) = 0$.

Solution. By applying the Mean Value Theorem to f on the interval $[a, x_0]$, we have

$$\frac{f(x_0) - f(a)}{x_0 - a} = f'(c_1) \quad \text{for some } c_1 \in (a, x_0).$$

By applying the Mean Value Theorem to f on the interval $[x_0, b]$, we have

$$\frac{f(b) - f(x_0)}{b - x_0} = f'(c_2) \quad \text{for some } c_2 \in (x_0, b).$$

Since $(a, f(a)), (x_0, f(x_0)), (b, f(b))$ lie on the same straight line, we have

$$\frac{f(x_0) - f(a)}{x_0 - a} = \frac{f(b) - f(x_0)}{b - x_0}.$$

Note that f' is continuous and differentiable on $[c_1, c_2]$. Another application of the Mean Value Theorem implies that

$$f''(c) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = 0 \quad \text{for some } c \in (c_1, c_2).$$

□

22. The equation $\ln x = x - 2$ has two solutions. Approximate them using Newton's Method. What happens if $x_1 := \frac{1}{2}$ is the initial point?

Solution. Let $f(x) = \ln x - x + 2$. It is clearly twice differentiable on $(0, \infty)$ with $f'(x) = \frac{1}{x} - 1$ and $f''(x) = -\frac{1}{x^2}$.
First, we will apply Newton's Method (and its proof) to the intervals

$$I_1 := [0.14, 0.16] \quad \subseteq \quad I_1' := [0.1, 0.2].$$

Note that $f(0.14) \approx -0.1061 < 0$ and $f(0.16) \approx 0.0074 > 0$. The Intermediate Value Theorem implies that there is $r_1 \in I_1$ such that $f(r_1) = 0$. Moreover,

$$m_1 := \min_{x \in I_1'} |f'(x)| = \frac{1}{0.2} - 1 = 4, \qquad M_1 := \max_{x \in I_1'} |f''(x)| = \frac{1}{0.1^2} = 100$$

Then $K_1 := M_1/2m_1 = 25/2$ satisfies $1/K_1 = 0.08 > \text{length}(I_1) = 0.02$. Take $\delta_1 = 0.02 \in (0, 1/K_1)$. So the interval $I_1^* := (r_1 - \delta_1, r_1 + \delta_1)$ satisfies $I_1 \subseteq I_1^* \subseteq I_1'$. Hence, by Newton's Method and its proof, for any $x_1 \in I_1^*$, the sequence (x_n) defined by

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for all } n \in \mathbb{N},$$

belongs to I_1^* and (x_n) converges to r_1. Since $I_1 \subseteq I_1^*$, we can pick any $x_1 \in I_1$ as the initial point. For example,

$$x_1 = 0.14, \quad x_2 \approx 0.1573, \quad x_3 \approx 0.1586, \quad x_3 \approx 0.1586, \quad \cdots$$

Next, we will apply Newton's Method (and its proof) to the intervals

$$I_2 := [3, 4] \quad \subseteq \quad I_2' := [2, 5].$$

Note that $f(3) \approx 0.0986 > 0$ and $f(4) \approx -0.6137 < 0$. The Intermediate Value Theorem implies that there is $r_2 \in I_2$ such that $f(r_2) = 0$. Moreover,

$$m_2 := \min_{x \in I_2'} |f'(x)| = 1 - \frac{1}{5} = 0.8 \qquad M_2 := \max_{x \in I_2'} |f''(x)| = \frac{1}{2^2} = 0.25.$$

Then $K_2 := M_2/2m_2 = 5/32$ satisfies $1/K_2 = 32/5 > \text{length}(I_2) = 1$. Take $\delta_2 = 1 \in (0, 1/K_2)$. So the interval $I_2^* := (r_2 - \delta_2, r_2 + \delta_2)$ satisfies $I_2 \subseteq I_2^* \subseteq I_2'$. Hence, by Newton's Method and its proof, for any $x_1 \in I_2^*$, the sequence (x_n) defined by

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for all } n \in \mathbb{N},$$

belongs to I_2^* and (x_n) converges to r_2. Since $I_2 \subseteq I_2^*$, we can pick any $x_1 \in I_2$ as the initial point. For example,

$$x_1 = 3, \quad x_2 \approx 3.1479, \quad x_3 \approx 3.1462, \quad x_3 \approx 3.1462, \quad \cdots$$

If $x_1 = \frac{1}{2}$ is the initial point, then $x_2 = \ln 2 - 1 < 0$. The Newton's Method cannot proceed because $f(x_2)$ is not defined.

□