§9.4 Series of Functions

Def 9.4.1

If (f_n) is a seq. of functions defined on $D \subseteq \mathbb{R}$ (with $\mathbb{R}$-value), then the sequence of partial sums (S_n) of the infinity series of functions $\sum f_n$ is defined by

$$S_n(x) = \sum_{k=1}^{n} f_k(x), \quad \forall x \in D$$

- If (S_n) converges to a function f on D, then we say that the infinite series of functions $\sum f_n$ converges to f on D.

(usually write $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $f = \sum_{n=1}^{\infty} f_n$, or $f = \sum f_n$)

- If $\sum |f_n(x)|$ converges $\forall x \in D$, then we say that $\sum f_n$ is absolutely convergent on D.
- If $S_n \rightrightarrows f$ (uniformly) on D, then we say that $\sum f_n$ is uniformly convergent on D, or $\sum f_n$ converges to f uniformly on D

Using $S_n \rightrightarrows f \Leftrightarrow \sum f_n$ converges to f uniformly, Thm 8.2.2, Thm 8.2.3 & Thm 8.2.4 imply the following theorems immediately:

Thm 9.4.2 If
- f_n continuous on D, $\forall n \in \mathbb{N}$
- $\sum f_n$ converges to f uniformly on D

Then f is continuous on D

(Pf : Applying Thm 8.2.2 to $S_n \rightrightarrows f$.)

Thm 9.4.3 If
- $f_n \in R[a,b]$, $\forall n \in \mathbb{N}$ $\quad (a<b \in \mathbb{R})$
- $\sum f_n$ converges to f uniformly on $[a,b]$

Then $f \in R[a,b]$ and $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$

$$\left(\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n \right)$$

(Pf : Applying Thm 8.2.4 to $S_n \rightrightarrows f$.)

Thm 9.4.4 If
- $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$ $\quad (a<b \in \mathbb{R})$,
- f_n' exists on $[a,b]$, $\forall n \in \mathbb{N}$,
- $\exists x_0 \in [a,b]$ s.t. $\sum f_n(x_0)$ converges,
- $\sum f_n'$ converges uniformly on $[a,b]$.

Then $\exists f : [a,b] \to \mathbb{R}$ such that
- $\sum f_n$ converges to f uniformly on $[a,b]$,
- f' exists and $f' = \sum_{n=1}^{\infty} f_n'$

(Pf : Applying Thm 8.2.3 to S_n with S_n' converges uniformly etc.)

Tests for Uniform Convergence

Thm 9.4.5 (Cauchy Criterion)

$\sum f_n$ is uniformly convergent on D $\Leftrightarrow$

$\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ such that

if $m > n \geq K(\varepsilon)$, then $|f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon$, $\forall x \in D$.

(Pf: Applying Cauchy Criterion for Uniform Convergence (Thm 8.1.10) to S_n and observing that

$$S_m(x) - S_n(x) = f_{n+1}(x) + \cdots + f_m(x) .)$$

Thm 9.4.6 (Weierstrass M-Test)

If $\begin{cases} \cdot\ |f_n(x)| \leq M_n, \ \forall x \in D, \ \forall n \in \mathbb{N} \\ \cdot\ \sum M_n \text{ is convergent} \end{cases}$

then $\sum f_n$ is uniformly convergent on D

Pf: $\sum M_n$ convergent & $M_n \geq 0 \Rightarrow$

$\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ such that

if $m > n \geq K(\varepsilon)$, then $M_{n+1} + \cdots + M_m < \varepsilon$. (Thm 3.7.4)

Hence $|f_{n+1}(x) + \cdots + f_m(x)| \leq M_{n+1} + \cdots + M_m < \varepsilon$.

Cauchy Criterion (Thm 9.4.5) $\Rightarrow$ $\sum f_n$ converges uniformly on D ※

Power Series

Def 9.4.7 If $f_n(x) = a_n(x-c)^n$, a_n & $c \in \mathbb{R}$, $\forall n = 0,1,2,\cdots$

then $\sum f_n(x) = \sum a_n(x-c)^n$

is called a <u>power series around $x=c$</u>.

<u>Remarks</u>: • Power series usually starts with $n=0$ (instead of $n=1$):

$$\sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

• $\sum a_n x^n$ may <u>not</u> defined over all of $\mathbb{R}$:

(i) $\sum_{n=0}^{\infty} n!\, x^n$ converges only for $x=0$, (Ex!)

(ii) $\sum_{n=0}^{\infty} x^n$ converges for $|x|<1$, (geometric series)

(iii) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges $\forall x \in \mathbb{R}$, (exponential function)

Hence there is a need to determine

the <u>set</u> on which $\sum a_n x^n$ converges.

In the following, we consider the case that "$c=0$".

This is no loss of generality as the "translation $y = x-c$"

redues $\sum a_n (x-c)^n$ to $\sum a_n y^n$.

<u>Recall</u>: (Def 3.4.10 & Thm 3.4.11)

For (x_n) a <u>bounded</u> seq., <u>limit superior</u> of (x_n):

$$\limsup x_n \overset{\text{def}}{=} \inf\{v \in \mathbb{R} : v < x_n \text{ for finitely many } n\}$$
$$= \inf\{v \in \mathbb{R} : x_n \leq v \text{ for sufficiently large } n\}$$

And (i) If $v > \limsup x_n$, then

$x_n \leq v$ for sufficiently large n,

i.e. $\exists K(v) \in \mathbb{N}$ s.t. if $n \geq K(v)$, then $x_n \leq v$.

(ii) If $w < \limsup x_n$, then $\exists$ <u>infinitely</u> many $n \in \mathbb{N}$

s.t. $w \leq x_n$.

<u>Def 9.4.8</u> Let
- $\sum a_n x^n$ be a power series, and
- $\rho = \begin{cases} \limsup(|a_n|^{\frac{1}{n}}), & \text{if } (|a_n|^{\frac{1}{n}}) \text{ is a bdd seq.} \\ +\infty, & \text{otherwise} \end{cases}$

Then
- the <u>radius of convergence</u> of $\sum a_n x^n$ is defined by

$$R = \frac{1}{\rho} = \begin{cases} 0, & \text{if } \rho = +\infty \\ \dfrac{1}{\limsup |a_n|^{\frac{1}{n}}}, & \text{otherwise } \left(\text{including } R = +\infty \text{ when } \limsup |a_n|^{\frac{1}{n}} = 0\right) \end{cases}$$

- The <u>interval of convergence</u> is the <u>open</u> interval $(-R, R)$

Thm 9.4.9 (Cauchy-Hadamard Theorem)

If R is the radius of convergence of $\sum a_n x^n$, then

$$\sum a_n x^n \text{ is } \begin{cases} \bullet \text{ absolutely convergent} & \text{if } |x| < R, \\ \bullet \text{ divergent} & \text{if } |x| > R. \end{cases}$$

Remark : No conclusion for $|x| = R$:

(i) $\sum x^n$: $\rho = \limsup |a_n|^{\frac{1}{n}} = \limsup 1 = 1$

$\Rightarrow R = \frac{1}{\rho} = 1.$

$$\begin{cases} x = 1 : & \sum x^n = 1 + 1 + 1 + \cdots \text{ is divergent} \\ x = -1 : & \sum x^n = -1 + 1 - 1 + 1 - \cdots \text{ is divergent.} \end{cases}$$

(ii) $\sum \frac{1}{n} x^n$: $\rho = \limsup |a_n|^{\frac{1}{n}} = \limsup \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1$ (Ex!)

$\Rightarrow R = \frac{1}{\rho} = 1$

$$\begin{cases} x = 1 : & \sum \frac{1}{n} x^n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \text{ is divergent} \\ x = -1 : & \sum \frac{1}{n} x^n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \text{ is convergent.} \end{cases}$$

(iii) $\sum \frac{1}{n^2} x^n$: $\rho = \limsup |a_n|^{\frac{1}{n}} = \limsup \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = 1$ (Ex!)

$\Rightarrow R = \frac{1}{\rho} = 1$

$$\begin{cases} x = 1 : & \sum \frac{1}{n^2} x^n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \text{ is convergent.} \\ x = -1 : & \sum \frac{1}{n^2} x^n = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \text{ is convergent.} \end{cases}$$

Pf of Cauchy-Hadamard Thm:

- $R=0$ and $R=+\infty$ leave as exercises.

Assume $0<R<+\infty$.

Clearly $\sum a_n x^n$ converges for $x=0$.

Consider $0<|x|<R$,

then $\exists\ 0<c<1$ such that $|x|<cR \ (=\frac{c}{\rho})$

Therefore $\rho|x| = \limsup\left(|a_n|^{\frac{1}{n}}|x|\right) < c$.

$\Rightarrow \exists K\in\mathbb{N}$ such that

if $n\geq K$, then $|a_n|^{\frac{1}{n}}|x| \leq c$.

$$\Rightarrow \quad |a_n x^n| \leq c^n, \ \forall n\geq K$$

Since $0<c<1$, $\sum c^n$ is convergent.

By Comparison Test (Thm 3.7.7), $\sum|a_n x^n|$ is convergent

i.e. $\sum a_n x^n$ is absolutely convergent.

This proves the 1st part.

If $|x|>R=\frac{1}{\rho}$, then $\rho = \limsup|a_n|^{\frac{1}{n}} > \frac{1}{|x|}$.

$\Rightarrow |a_n|^{\frac{1}{n}} > \frac{1}{|x|}$ for infinitely many $n\in\mathbb{N}$

i.e. $|a_n x^n|>1$ for infinitely many $n\in\mathbb{N}$

and hence $a_n x^n \not\to 0$ $\therefore$ $\sum a_n x^n$ is divergent. ※

<u>Remarks</u>: (i) $\boxed{\text{If } \lim \left|\frac{a_n}{a_{n+1}}\right| \text{ exists, then radius of convergence} = \lim \left|\frac{a_n}{a_{n+1}}\right|}$.

(Notes: • it is the reciprocal of $\left|\frac{a_{n+1}}{a_n}\right|$ in ratio test. (Ex 9.4.5)

• $\left|\frac{a_n}{a_{n+1}}\right| \to \infty$ is included)

When exists, it is usually easier to calculate:

(a) $\sum x^n$: $a_n = 1, \forall n$. $\left|\frac{a_n}{a_{n+1}}\right| \equiv 1 \to 1$ as $n \to \infty$

$\therefore R = 1$

(b) $\sum \frac{1}{n} x^n$: $a_n = \frac{1}{n}, \forall n$. $\left|\frac{a_n}{a_{n+1}}\right| = \frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{n+1}{n} \to 1$ as $n \to \infty$

$\therefore R = 1$

(c) $\sum \frac{1}{n^2} x^n$: $a_n = \frac{1}{n^2}, \forall n$. $\left|\frac{a_n}{a_{n+1}}\right| = \frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}} = \left(\frac{n+1}{n}\right)^2 \to 1$ as $n \to \infty$

$\therefore R = 1$

(ii) If one can choose $0 < c < 1$ independent of $x \in (-R, R)$, then one get uniform convergence. (Ex!)

<u>Thm 9.4.10</u>: Let
- • R = radius of convergence of $\sum a_n x^n$
- • $[a,b] \subset (-R,R)$ be a <u>closed and bounded</u> interval.

Then $\sum a_n x^n$ <u>converges uniformly on</u> $[a,b]$.

Remark
- $R=+\infty$ included, and hence we need the assumption that $[a,b]$ is bounded.
- $R=0$ is excluded as $(-0,0)=\emptyset$. (although $\sum a_n x^n$ converges for $x=0$)

Pf of Thm 9.4.10: Since $[a,b]\subset(-R,R)$, $\exists\ 0<c<1$ such that

$$-cR<a \quad \text{and} \quad b<cR. \quad (\text{Note: } c \text{ depends only on } a,b)$$

Therefore $\forall x\in[a,b]$, $|x|<cR$.

By argument in the proof of Cauchy-Hadamard Thm, we have

$\exists K\in\mathbb{N}$ s.t. $|a_n x^n|\le c^n$, $\forall n\ge K$ (Ex! use $0<c_1<c$ s.t. $-c_1R<a<b<c_1R$ to find a K indep. of x)

Since $\sum c^n$ is convergent, Weierstrass M-Test (Thm 9.4.6)

$\Rightarrow$ ($\sum_{n=K}^{\infty} a_n x^n$ and hence) $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[a,b]$. #

Thm 9.4.11
- The limit of power series is continuous on the interval of convergence.
- A power series can be integrated term-by-term over any closed and bounded interval contained in the interval of convergence.

Pf: • $\forall x \in (-R, R)$, choose a closed & bounded interval $[a,b]$ s.t. $x \in [a,b] \subset (-R, R)$. Then on $[a,b]$,

$\sum a_n x^n$ converges uniformly. (Thm 9.4.10)

Thm 9.4.2 $\Rightarrow$ $\sum_{n=1}^{\infty} a_n x^n$ is continuous on $[a,b]$ and hence at x

Since $x \in (-R, R)$ is arbitrary, $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R, R)$.

• For any closed and bounded interval $[a,b] \subset (-R, R)$, $\sum a_n x^n$ converges uniformly on $[a,b]$

and hence Thm 9.4.3 $\Rightarrow$

$$\int_a^b \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \int_a^b a_n x^n . \quad ※$$

<u>Thm 9.4.12</u> (<u>Differentiation Thm</u>)

A power series can be <u>differentiated term-by-term</u> within the <u>interval of convergence</u>. In fact, if R = radius of convergence of $\sum a_n x^n$

and $f(x) = \sum_{n=0}^{\infty} a_n x^n$, for $|x| < R$,

then <u>the radius of convergence of $\sum_{n=0}^{\infty} n a_n x^{n-1} = R$</u>,

and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, for $|x| < R$

Pf: Since $n^{\frac{1}{n}} \to 1$, the seq. $\left(|(n+1)a_{n+1}|^{\frac{1}{n}}\right)$ is bounded

$\Leftrightarrow$ the seq $\left(|a_n|^{\frac{1}{n}}\right)$ is bounded

unbounded case: $R=0 \Leftrightarrow$ Radius of convergence of $\sum n a_n x^{n-1} = 0$

bounded case:

Radius of convergence of $\sum n a_n x^{n-1} = \limsup |(n+1)\, a_{n+1}|^{\frac{1}{n+1}}$

$= \limsup |n a_n|^{\frac{1}{n}} = \limsup \left(n^{\frac{1}{n}} |a_n|^{\frac{1}{n}}\right)$

$= \limsup |a_n|^{\frac{1}{n}}$ (since $n^{\frac{1}{n}} \to 1$)

$= R$.

Hence Radius of convergence of $\sum n a_n x^{n-1}$

$=$ Radius of convergence of $\sum a_n x^n$.

Note that $\sum a_n x^n$ converges for $x=0$

Now $\forall\, x \in (-R,R)$, choose $0<a<R$ such that $|x|<a$.

Then
- $[-a,a]$ is closed and bounded,
- $[-a,a] \subset (-R,R)$ and
- $0 \in [-a,a]$ s.t. $\sum a_n x^n$ converges at $x=0$.

Using Thm 9.4.10, Thm 8.2.3 and note that

- $(a_n x^n)' = n a_n x^{n-1}$

- $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} (a_n x^n)'$ converges uniformly on $[-a, a]$,

we have $\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ on $[-a, a]$

and in particular for x.

Since $x \in (-R, R)$ is arbitrary, we have

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \forall x \in (-R, R)$$

<u>Remarks</u>: (i) Differentiation Thm 9.4.12 makes no conclusion for $|x| = R$:

e.g. $\sum \frac{1}{n^2} x^n$ converges for $|x| = 1$ $(= R)$

but $\left(\sum \frac{1}{n^2} x^n\right)' = \sum \frac{1}{n} x^{n-1}$ $\begin{cases} \text{converges at } x = -1 \\ \text{diverges at } x = 1. \end{cases}$

(ii) Repeated application of Thm 9.4.12 $\Rightarrow$

$\forall k \in \mathbb{N}$,

$$\boxed{\left(\sum_{n=0}^{\infty} a_n x^n\right)^{(k)} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}}$$