Cor 9.2.9 $\begin{cases} \bullet\ x_n \neq 0, \ \forall n = 1, 2, 3, \cdots \\ \bullet\ a = \lim\limits_{n\to\infty} n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right) \text{ exists} \end{cases}$

Then $\begin{cases} \bullet\ a > 1 \Rightarrow \sum x_n \text{ is absolutely convergent} \\ \bullet\ a < 1 \Rightarrow \sum x_n \text{ is not absolutely convergent} \end{cases}$

Pf: Omitted

Egs 9.2.10

(a) Raabe's Test for $\sum \frac{1}{n^p}$:

$$a = \lim_{n\to\infty} n\left(1 - \left|\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}\right|\right) = \lim_{n\to\infty} n\left(1 - \frac{n^p}{(n+1)^p}\right)$$

$$= \lim_{n\to\infty} n\left(1 - \frac{1}{(1+\frac{1}{n})^p}\right) = \lim_{n\to\infty}\left[\frac{(1+\frac{1}{n})^p - 1}{\frac{1}{n}} \cdot \frac{1}{(1+\frac{1}{n})^p}\right]$$

Clearly $\lim\limits_{n\to\infty} \frac{(1+\frac{1}{n})^p - 1}{\frac{1}{n}} = \frac{d}{dx}\Big|_{x=1} x^p = p$ (Thm 8.3.13)

$\therefore \quad a = p \cdot 1 = p.$

By Cor 9.2.9 to Raabe's Test (or just call it Raabe's Test),

$p > 1 \Rightarrow \sum \frac{1}{n^p}$ is (absolutely) convergent

$p < 1 \Rightarrow \sum \frac{1}{n^p}$ is not (absolutely) convergent

(hence divergent (as $\frac{1}{n^p} > 0, \forall n$))

However, result for $p = 1$ cannot be deduced from Raabe's Test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Easy to check:

$$\begin{cases} \bullet \left|\dfrac{x_{n+1}}{x_n}\right| = \dfrac{\frac{n+1}{(n+1)^2+1}}{\frac{n}{n^2+1}} = \dfrac{n+1}{n} \cdot \dfrac{n^2+1}{(n+1)^2+1} \to 1, \text{ and} \\ \bullet\ n\left(1 - \left|\dfrac{x_{n+1}}{x_n}\right|\right) = n \cdot \left(1 - \dfrac{n+1}{n} \cdot \dfrac{n^2+1}{(n+1)^2+1}\right) \end{cases}$$

$$= \frac{n^2+n-1}{(n+1)^2+1} \longrightarrow 1 \quad \text{as } n \to \infty$$

$\therefore$ Both Cor 9.2.5 and Cor 9.2.2 cannot be applied.

But
$$\left|\frac{x_{n+1}}{x_n}\right| - 1 = \frac{n+1}{n} \frac{n^2+1}{(n+1)^2+1} - 1 = \frac{(n+1)(n^2+1) - n[(n+1)^2+1]}{n[(n+1)^2+1]}$$

$$= -\frac{n^2+n-1}{n[(n+1)^2+1]} = -\frac{1}{n} \cdot \frac{n^2+n-1}{n^2+2n+2} \geq -\frac{1}{n}$$

$$\therefore \left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{1}{n}, \quad \forall n \geq 1 \quad \left(\begin{array}{l} a = 1 \leq 1 \\ \& \ K = 1 \in \mathbb{N} \end{array}\right)$$

Raabe's Test (Thm 9.2.8) $\Rightarrow \sum x_n$ is not absolutely convergent.

Remarks: (i) "Limiting form" of Raabe's Test (Cor 9.2.9) doesn't apply, but Raabe's Test (Thm 9.2.8) applies.

(ii) Integral Test or Limit Comparison Test work for this example.

§9.3 Tests for Nonabsolute Convergence

Def 9.3.1 • $x_n \neq 0$, $\forall n \in \mathbb{N}$

Then • the sequence (x_n) is said to be alternating

$$\text{if } (-1)^{n+1} x_n > 0 \ (\text{or} < 0) \quad \forall n \in \mathbb{N}$$

• in this case, the series $\sum x_n$ is called an alternating series.

e.g. If $z_n > 0$, then $x_n = (-1)^{n+1} z_n$ and $x_n = (-1)^n z_n$ are alternating

(explicit eg: $z_n = \frac{1}{n} > 0$, $(x_n) = \left((-1)^{n+1} z_n\right) = (1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots)$ is alternating)

Thm 9.3.2 Let • $z_n > 0$ and decreasing $(z_{n+1} \leq z_n)$ $\forall n \in \mathbb{N}$

• $\lim_{n \to \infty} z_n = 0$

Then the alternating series $\sum (-1)^{n+1} z_n$ is convergent

Pf: Consider partial sum

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} z_k = z_1 - z_2 + z_3 - z_4 + \cdots + z_{2n-1} - z_{2n}$$

Then $S_{2(n+1)} - S_{2n} = z_{2n+1} - z_{2n+2} \geq 0$, since z_n is decreasing

$\therefore$ (S_{2n}) is increasing (in n).

Also $z_1 - S_{2n} = z_2 - z_3 + z_4 - z_5 + \cdots + z_{2n-2} - z_{2n-1} + z_{2n} > 0$

$$\Big(\underbrace{z_2 - z_3}_{\geq 0} \quad \underbrace{z_4 - z_5}_{\geq 0} \quad \underbrace{z_{2n-2} - z_{2n-1}}_{\geq 0} \quad \underbrace{z_{2n}}_{> 0} \Big)$$

$\therefore$ (S_{2n}) is bounded above by z_1

By Monotone Convergence Thm (Thm 3.3.2), $\exists\, S \in \mathbb{R}$ s.t.

$$S_{2n} \to S \quad \text{as } n \to \infty.$$

Together with $z_n \to 0$ as $n \to \infty$, we have

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t.

if $n \geq K$, then $\begin{cases} \bullet\ |S_{2n} - S| < \frac{\varepsilon}{2}, \text{ and} \\ \bullet\ (0<)\, z_{2n+1} < \frac{\varepsilon}{2}. \end{cases}$

Then $$|S_{2n+1} - S| = |z_{2n+1} + S_{2n} - S|$$

$$\leq |z_{2n+1}| + |S_{2n} - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore$ $S_{2n+1} \to S$ as $n \to \infty$.

Combining with $S_{2n} \to S$ as $n \to \infty$, we have

$$\lim S_n = S$$

$\therefore$ $\sum (-1)^{n+1} z_n$ is convergent. ※

egs: By Thm 9.3.1, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

is convergent

(Note: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$ is divergent by integral Test)

eg 9.2.7 (d)

The Dirichlet and Abel Tests

Thm 9.3.3 (Abel's Lemma)

Let
- (x_n), (y_n) be sequences in $\mathbb{R}$, and
- $\begin{cases} s_0 = 0, & \& \\ s_n = \sum_{k=1}^{n} y_k, & n = 1, 2, 3, \dots \end{cases}$

Then for $m > n$,

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k .$$

Pf : $\displaystyle\sum_{k=n+1}^{m} x_k y_k = \sum_{k=n+1}^{m} x_k (s_k - s_{k-1})$

$$= x_m (s_m - s_{m-1}) + x_{m-1}(s_{m-1} - s_{m-2}) + \dots + x_{n+1}(s_{n+1} - s_n)$$

$$= x_m s_m - (x_m - x_{m-1}) s_{m-1} - (x_{m-1} - x_{m-2}) s_{m-2} - \dots$$
$$- (x_{n+2} - x_{n+1}) s_{n+1} - x_{n+1} s_n$$

$$= (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k$$

\#

Thm 9.3.4 (**Dirichlet's Test**)

If $\begin{cases} \bullet\ (x_n) \text{ decreasing } \& \lim x_n = 0 \\ \bullet\ (S_n = \sum_{k=1}^{n} y_k) \text{ are bounded}, \end{cases}$

then $\sum x_n y_n$ is convergent.

Pf: (S_n) bdd $\Rightarrow \exists B>0$ s.t. $|S_n| \le B$, $\forall n \in \mathbb{N}$.

Then Abel's Lemma (Thm 9.3.3) $\Rightarrow$ for $m>n$,

$$\left|\sum_{k=n+1}^{m} x_k y_k\right| \le |x_m S_m - x_{n+1} S_n| + \sum_{k=n+1}^{m-1} |x_k - x_{k+1}||S_k|$$

$$\le (x_m + x_{n+1})B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1})B \quad \left(\text{since } x_n \text{ decreasing},\ x_k - x_{k+1} \ge 0\right)$$

$$= B[(x_m + x_{n+1}) + (x_{n+1} - x_m)]$$

$$= 2x_{n+1}B \longrightarrow 0 \quad \text{as } n \to \infty$$

($\therefore \forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. if $m > n \ge K$, $|\sum_{k=n+1}^{m} x_k y_k| < \varepsilon$)

By Cauchy Criterion (Thm 3.7.4), $\sum x_n y_n$ is convergent. #

Thm 9.3.5 (**Abel's Test**)

If $\begin{cases} \bullet\ (x_n) \text{ convergent monotone sequence} \\ \bullet\ \sum y_n \text{ convergent} \end{cases}$

Then $\sum x_n y_n$ is also convergent.

(multiplying convergent monotone coefficients to a convergent series results in a convergent series.)

Pf: Case 1: (x_n) decreasing & $\lim x_n = x$

Let $u_n = x_n - x$, $\forall n \in \mathbb{N}$.

Then (u_n) decreasing & $u_n \to 0$.

Now $\sum y_n$ converges $\Rightarrow$ partial sum of $\sum y_n$ are bounded

$\therefore$ Dirichlet's Test (Thm 9.3.4) $\Rightarrow$ $\sum u_n y_n$ is convergent.

Hence $\sum x_n y_n = \sum (u_n + x) y_n = \sum u_n y_n + x \sum y_n$

is also convergent.

Case 2 (x_n) increasing, $x = \lim x_n$.

Similar argument as in case 1 by considering

$v_n = x - x_n$, $\forall n \in \mathbb{N}$ instead of u_n.

※

Eg 9.3.6 (a) Recall $2\sin\frac{1}{2}x\,(\cos x + \cdots + \cos nx) = \sin(n+\frac{1}{2})x - \sin\frac{1}{2}x$ (Ex)

If x is fixed and $x \neq 2k\pi$, $\forall k = \cdots -2, -1, 0, 1, 2, \cdots$

Then $\underbrace{|\cos x + \cdots + \cos nx|}_{\text{partial sum of } \sum \cos nx} = \dfrac{|\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x|}{2|\sin\frac{1}{2}x|} \leq \underbrace{\dfrac{1}{|\sin\frac{1}{2}x|}}_{\text{bound indep. of } n}$, $\forall n \in \mathbb{N}$

$\therefore$ For a fixed $x \neq 2k\pi$, Dirichlet's Test $\Rightarrow$

$\sum_{n=1}^{\infty} a_n \cos nx$ converges, provided $\begin{cases} (a_n) \text{ is decreasing } \& \\ \lim a_n = 0 \end{cases}$.

(b) Similarly, from

$$2\left(\sin\tfrac{1}{2}x\right)(\sin x + \cdots + \sin nx) = \cos\tfrac{1}{2}x - \cos\left(n+\tfrac{1}{2}\right)x \quad , \quad \forall n \in \mathbb{N}$$

we have, for $x \neq 2k\pi$,

$$\underbrace{|\sin x + \cdots + \sin nx|}_{\text{partial sum of } \sum \sin nx} \leq \underbrace{\frac{1}{|\sin\frac{1}{2}x|}}_{\text{bound indep. of } n} \qquad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} a_n \sin nx \text{ converges for } x \neq 2k\pi$$

provided $\begin{cases} (a_n) \text{ decreasing and} \\ \lim a_n = 0 . \end{cases}$