<u>Claim</u>: Let $f > 0$, on $[1,\infty)$; $f \in R[1,b]$, $\forall b > 1$, then

$\lim_{n\to\infty} \int_1^n f(t)\,dt$ exists $\iff$ $\lim_{b\to\infty} \int_1^b f(t)\,dt$ exists.

Pf: ($\Rightarrow$) Assume $\lim_{n\to\infty} \int_1^n f$ exists.

$\forall b > 1$, $\exists n \in \mathbb{N}$ s.t. $\quad n \le b < n+1$

(in fact n = largest integer $\le b$.)

Since $f > 0$,

$$\int_1^n f(t)\,dt \le \int_1^b f(t)\,dt \le \int_1^{n+1} f(t)\,dt$$

Since $b \to \infty \Rightarrow n \to \infty$, and

$$\lim_{n\to\infty} \int_1^n f(t)\,dt = \lim_{n\to\infty} \int_1^{n+1} f(t)\,dt$$

$\therefore \quad \lim_{b\to\infty} \int_1^b f(t)\,dt$ exists and $= \lim_{n\to\infty} \int_1^n f(t)\,dt$

($\Leftarrow$) Assume $\lim_{b\to\infty} \int_1^b f$ exists.

Then subseq $\int_1^n f$ has limit & equals $\lim_{b\to\infty} \int_1^b f$.

This completes the proof of the integral test.

Egs 9.2.7

(a) Recall Eg 3.7.2(c)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$ is (absolutely) convergent. $\left(\frac{1}{n(n+1)} > 0\right)$

Using Limit Comparison Test II (Thm 9.2.1)

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$$

$\Rightarrow$ $\sum \frac{1}{n^2}$ is (absolutely) convergent.

(b) However, Root Test (Thm 9.2.2) doesn't apply to $\sum \frac{1}{n^2}$ (in fact $\sum \frac{1}{n^p}$, $\forall p > 0$):

- $\left(\frac{1}{n^p}\right)^{\frac{1}{n}} < 1$, and
- $\left|\frac{1}{n^p}\right|^{1/n} = \frac{1}{(n^{\frac{1}{n}})^p} \longrightarrow 1$ since $n^{\frac{1}{n}} \to 1$

$\therefore$ both conditions in part (a) & part (b) don't hold.

And the Cor 9.2.3 cannot be applied too.

$$\left(r = \lim_{n\to\infty} \left|\frac{1}{n^p}\right|^{\frac{1}{n}} = 1.\right)$$

(c) Ratio Test (Thm 9.2.4) and its Cor 9.25 also don't work

for $\sum \frac{1}{n^p}$:

$$\left| \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} \right| = \frac{n^p}{(n+1)^p} = \frac{1}{(1+\frac{1}{n})^p} \to 1$$

$r=1$, no information from Ratio test!

(d) On the other hand, Integral Test (Thm 9.2.6) works for $\sum \frac{1}{n^p}$:

Let $f(t) = \frac{1}{t^p}$, $t \geq 1$.

Then $f(t) > 0$ and decreasing.

$$\lim_{n\to\infty} \int_1^n \frac{1}{t^p}\,dt = \begin{cases} \lim_{n\to\infty} \left(\log(n) - \log 1\right), & p=1 \\ \lim_{n\to\infty} \left[\frac{t^{1-p}}{1-p}\right]_1^n, & p \neq 1 \end{cases}$$

(same as $b \to \infty$)

Since $\log(n) \to +\infty$, $\int_1^\infty \frac{1}{t}\,dt$ doesn't exist

$$\lim_{n\to\infty} \frac{1}{1-p}\left(\frac{1}{n^{p-1}} - 1\right) = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ +\infty, & \text{if } p < 1 \end{cases}$$

$\therefore \int_1^\infty \frac{1}{t^p}\,dt$ $\begin{cases} \text{exists} & \text{if } p > 1 \\ \text{doesn't exist} & \text{if } p < 1. \end{cases}$

Altogether, $\sum \frac{1}{n^p}$ $\begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$. $\#$

Thm 9.2.8 (Raabe's Test) Suppose $x_n \neq 0$, $\forall n = 1, 2, 3, \cdots$

(a) If $\exists$ $a > 1$ and $K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n}, \quad \forall n \geq K$$

(Note: This condition allows $\lim \left|\frac{x_{n+1}}{x_n}\right| = 1$)

then $\sum x_n$ is absolutely convergent

(b) If $\exists$ $a \leq 1$ and $K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n} \quad \forall n \geq K$$

(Note: This condition allows $\lim \left|\frac{x_{n+1}}{x_n}\right| = 1$)

then $\sum x_n$ is not absolutely convergent.

Pf: Omitted