Cor 8.4.2 If C, S are the functions in Thm 8.4.1, then

(iii) $$\begin{cases} C'(x) = -S(x), \\ S'(x) = C(x) \end{cases} \qquad \forall x \in \mathbb{R}.$$

Moreover, C & S have derivatives of all orders

Pf: Easy

Cor 8.4.3 The functions C & S in Thm 8.4.1 satisfy the <u>Pythagorean Identity</u>: $(C(x))^2 + (S(x))^2 = 1, \quad \forall x \in \mathbb{R}$

Pf: Let $f(x) = (C(x))^2 + (S(x))^2$.

By Thm 8.4.1, f is differentiable &

$$f'(x) = 2C(x)C'(x) + 2S(x)S'(x)$$

$$= -2C(x)S(x) + 2S(x)C(x) = 0, \quad \forall x \in \mathbb{R}$$

$\Rightarrow f(x)$ is a constant function on $\mathbb{R}$.

$\Rightarrow f(x) \equiv f(0) = (C(0))^2 + (S(0))^2 = 1, \quad \forall x \in \mathbb{R}.$ ※

Thm 8.4.4 The functions C and S satisfying

$$(*)_C \begin{cases} C'' = -C \\ C(0) = 1 \\ C'(0) = 0 \end{cases} \quad \text{and} \quad (*)_S \begin{cases} S'' = -S \\ S(0) = 0 \\ S'(0) = 1 \end{cases}$$

are unique.

Pf: Omitted (similar argument as in the proof for exponential function E by using Taylor's Thm, but reduce to "two" terms instead of "one" because the equations are 2^{nd} order.)

Def 8.4.5 The unique functions C & S given in Thm 8.4.1 are called the <u>cosine function</u> and the <u>sine function</u> respectively, and denoted by

$$\cos x = C(x) \quad \& \quad \sin x = S(x)$$

Thm 8.4.6: If $f: \mathbb{R} \to \mathbb{R}$ satisfies $f''(x) = -f(x)$, $\forall x \in \mathbb{R}$, then $\exists$ real numbers α, β such that

$$f(x) = \alpha C(x) + \beta S(x), \quad \forall x \in \mathbb{R}.$$

Pf: Let $\alpha = f(0)$ & $\beta = f'(0)$.

And consider $g(x) = \alpha C(x) + \beta S(x)$, $\forall x \in \mathbb{R}$.

Then
- $g(0) = \alpha C(0) + \beta S(0) = \alpha = f(0)$
- $g'(x) = -\alpha S(x) + \beta C(x)$

  $\Rightarrow g'(0) = -\alpha S(0) + \beta C(0) = \beta = f'(0)$.
- $g''(x) = \alpha C''(x) + \beta S''(x) = -g(x)$

Hence the function $h = f - g$ satisfies

$$\begin{cases} h'' = f'' - g'' = -f - (-g) = -h \\ h(0) = f(0) - g(0) = 0 \\ h'(0) = f'(0) - g'(0) = 0 \end{cases}$$

Similarly argument as in the proof of Thm 8.4.4, we have $h(x) = 0$, $\forall x \in \mathbb{R}$.

$\therefore$ $f(x) = g(x) = \alpha C(x) + \beta S(x)$ $\forall x \in \mathbb{R}$. ※

Thm 8.4.7 The cosine $C(x)$ & sine $S(x)$ satisfy

(v) $C(-x) = C(x)$ & $S(-x) = -S(x)$ $\forall x \in \mathbb{R}$

(vi) $\begin{cases} C(x+y) = C(x)C(y) - S(x)S(y) \\ S(x+y) = S(x)C(y) + C(x)S(y) \end{cases}$ (compound angle formulae)

Pf: Omitted (Easy by Thm 8.4.4 & 8.4.6)

Thm 8.4.8 For $x \geq 0$,

(vii) $-x \leq S(x) \leq x$;

(viii) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1$;

(ix) $x - \frac{1}{6}x^3 \leq S(x) \leq x$;

(x) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

Pf: Omitted

Lemma 8.4.9
- $\exists$ a root γ of $C(x)$ in the interval $[\sqrt{2}, \sqrt{3})$.
- Moreover, $C(x) > 0 \ \forall \ x \in [0, \gamma)$.
- The number 2γ is the smallest positive root of $S(x)$.

Pf: By ineq. (X) in Thm 8.4.8

$$1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

we have $C(\sqrt{2}) \geq 0$ and

$$C(\sqrt{3}) \leq 1 - \frac{1}{2}(\sqrt{3})^2 + \frac{1}{24}(\sqrt{3})^4$$

$$= 1 - \frac{3}{2} + \frac{9}{24} = \frac{24 - 36 + 9}{24} = -\frac{1}{8} < 0$$

Intermediate value Thm $\Rightarrow$ $C(x) = 0$ for some $x \in [\sqrt{2}, \sqrt{3})$.

Let γ be the <u>smallest</u> such root of $C(x)$ in $[\sqrt{2}, \sqrt{3})$.

Then $\forall \ x \in [0, \gamma)$,

if $x \in [\sqrt{2}, \gamma)$, then $C(x) \neq 0$ by the choice of γ.

If $x \in [0, \sqrt{2})$, then $C(x) \geq 1 - \frac{1}{2}x^2 > 0$.

Therefore, continuity of $C(x)$ $\Rightarrow$ $C(x) > 0$, $\forall \ x \in [0, \gamma)$

Finally by Thm 8.4.7 (with $x = y$), $S(2x) = 2S(x)C(x)$

Therefore $S(2\gamma) = 2S(\gamma)C(\gamma) = 0$

$\therefore$ 2γ is a positive root of $S(x)$.

Now let 2δ = smallest positive root of $S(x)$.

Existence of δ follows from $S(0)=0$ & $S'(0)=1$

Suppose $\delta<\gamma$

Then $0 = S(2\delta) = 2S(\delta)C(\delta)$,

Since $C(x)>0$, $\forall x\in[0,\gamma)$, we have

$$S(2\cdot\tfrac{\delta}{2}) = S(\delta) = 0$$

which contradicts the definition (smallest) of δ.

Therefore $\delta=\gamma$. $\#$

Note: Of course, we can prove that $\gamma>\sqrt{2}$ as stated in the Textbook. But we need Ex 8.4.4 (not just Thm 8.4.8).

$\boxed{\text{Def 8.4.10} \quad \pi \overset{\text{def}}{=} 2\gamma = \text{smallest positive root of } S}$

Note: Thm 8.4.8 (x) $\Rightarrow$ $2.828 \le \pi \le 2\times\underbrace{\sqrt{6-2\sqrt{3}}}$ <3.185 (Ex!)

smallest positive root of $1-\frac{1}{2}x^2+\frac{1}{24}x^4$.

Thm 8.4.11

- C & S are 2π-periodic (have period 2π)

(xi) $C(x+2\pi) = C(x)$ & $S(x+2\pi) = S(x)$, $\forall x \in \mathbb{R}$

- $$\begin{cases} S(x) = C\left(\frac{\pi}{2} - x\right) = -C\left(x + \frac{\pi}{2}\right) \\ C(x) = S\left(\frac{\pi}{2} - x\right) = S\left(x + \frac{\pi}{2}\right) \end{cases} \quad \forall x \in \mathbb{R}$$

Pf Omitted.

Ch 9 Infinite Series

§9.1 Absolute Convergence

Recall Eg 3.7.6 (b) Harmonic series

$$\sum_{i=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \quad \text{is divergent}$$

(since partial sum $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is unbounded)

but Eg 3.7.6 (f) Alternating harmonic series

$$\sum_{i=1}^{n} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad \text{is convergent}$$

$\therefore$ A series $\sum x_n$ may be convergent, but the series $\sum |x_n|$ may be divergent

Def 9.1.1
- $\sum x_n$ is absolutely convergent if the series $\sum |x_n|$ is convergent
- $\sum x_n$ is conditionally convergent (or non-absolutely convergent) if $\sum x_n$ is convergent but $\sum |x_n|$ is divergent.

(i.e. conditionally convergent means convergent but not absolutely convergent)

Eg: Alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Thm 9.1.2 "Absolutely convergent" $\Rightarrow$ "convergent".

Pf: $\sum |x_n|$ convergent

$\Rightarrow \forall \varepsilon > 0, \exists M(\varepsilon) \in \mathbb{N}$ s.t. (Cauchy Criterion 3.7.4)

if $m > n \geq M(\varepsilon)$, then $|x_{n+1}| + \cdots + |x_m| < \varepsilon$

Let $S_n = x_1 + \cdots + x_n$ be the n^{th} partial sum of $\sum x_n$,

then $\forall m > n \geq M(\varepsilon)$,

$$|S_m - S_n| = |x_{n+1} + \cdots + x_m| \leq |x_{n+1}| + \cdots + |x_m| < \varepsilon.$$

$\therefore$ $\sum x_n$ is convergent. #

Grouping of Series

For a series of $\sum x_n$, one can construct many other series

$\sum y_k$ by "grouping the terms":

inserting parentheses that group together finitely many terms,

but keeping the order of the terms x_n fixed.

That is

$$y_1 = \sum_{j=1}^{n_1} x_j, \quad y_2 = \sum_{j=n_1+1}^{n_2} x_j, \cdots, y_k = \sum_{j=n_{k-1}+1}^{n_k} x_j, \cdots$$

$$(n_k < n_{k+1} \quad \forall k = 1, 2, \cdots \quad \& \quad n_0 = 0)$$

$\therefore \quad x_1 + x_2 + \cdots + x_n + \cdots$

$$= (x_1 + \cdots + x_{n_1}) + (x_{n_1+1} + \cdots + x_{n_2}) + (x_{n_2+1} + \cdots) + \cdots$$

$$= \quad y_1 \quad + \quad y_2 \quad + \quad y_3 \quad + \cdots$$

Eg: $1 - \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6} + \frac{1}{7}\right) - \frac{1}{8} + \left(\frac{1}{9} - \cdots + \frac{1}{13}\right) \cdots$

is a grouping the terms of the alternating harmonic series.

(i.e. $y_1 = 1$, $y_2 = -\frac{1}{2}$, $y_3 = \frac{1}{3} - \frac{1}{4}$, $y_4 = \frac{1}{5} - \frac{1}{6} + \frac{1}{7}$

$y_5 = -\frac{1}{8}$, $y_6 = \frac{1}{9} - \cdots + \frac{1}{13}$, $\cdots$)

Thm 9.1.3 $\sum x_n$ convergent

$\Rightarrow$ any series $\sum y_k$ obtained from it by grouping the terms is also convergent, & converges to the same value.

Pf: Let $S_n = n^{th}$ partial sum of $\sum x_n$

$t_k = k^{th}$ partial sum of $\sum y_k$.

If $y_k = \sum_{j=n_{k-1}+1}^{n_k} x_j$,

then $t_1 = y_1 = x_1 + \cdots + x_{n_1} = S_{n_1}$

$$t_2 = y_1 + y_2 = \sum_{j=1}^{n_1} x_j + \sum_{j=n_1+1}^{n_2} x_j = x_1 + \cdots + x_{n_2} = S_{n_2}$$

$\vdots$

$$t_k = S_{n_k}.$$

$\therefore$ (t_k) is a subseq. of (S_n)

Since $\sum x_n$ is convergent, $S_n \to S \left(= \sum_{n=1}^{\infty} x_n\right)$ as $n \to \infty$

$\therefore$ $t_k \to S$ as $k \to \infty$

i.e. $\sum y_k$ is convergent and converges to the same value as $\sum x_n$ $\#$

Remark: The converse of Thm 9.1.3 is not true.

Counterexample: Let $\sum x_n = 1 - 1 + 1 - 1 + 1 \cdots$

& $\sum y_k = (1-1) + (1-1) + (1-1) + \cdots$

Then $y_k = 0 \ \forall k \Rightarrow \sum y_k$ is convergent.

But original series $\sum x_n = 1 - 1 + 1 - 1 + 1 \cdots$ is divergent.

Rearrangement of series

(Not grouping any terms, but scrambling the order of the terms.)

Def 9.1.4 $\sum y_k$ is a rearrangement of $\sum x_n$, if $\exists$ a bijection (ie. one-to-one) $f: \mathbb{N} \to \mathbb{N}$ s.t. $$y_k = x_{f(k)} \quad \forall k \in \mathbb{N} = \{1, 2, 3, \cdots\}.$$

Remarks: (i) $\sum x_n$ is convergent $\not\Rightarrow$ $\sum y_k$ rearrangement is convergent

(Ex 9.1.3)

(ii) Riemann Thm: If $\sum x_n$ conditionally convergent,

then $\forall\, c \in \mathbb{R}$, $\exists$ a rearrangement $\sum y_k$ of $\sum x_n$ such that

$$\sum_{k=1}^{\infty} y_k = c \qquad \text{(Pf omitted)}$$

Thm 9.1.5 If $\sum x_n$ is absolutely convergent, then any rearrangement $\sum y_k$ of $\sum x_n$ converges to the same value.

Pf: $\sum x_n$ absolutely convergent $\Rightarrow$ $\sum x_n$ convergent.

Let $x = \sum_{n=1}^{\infty} x_n$, and $S_n = \sum_{k=1}^{n} x_k$.

Then $S_n \to x$ as $n \to \infty$

$\therefore$ $\forall\, \varepsilon > 0$, $\exists\, N_1 \in \mathbb{N}$ s.t.

$$\text{if } n \geq N_1, \quad |S_n - x| < \varepsilon.$$

On the other hand, $\sum |x_n|$ convergent

$\Rightarrow$ $\forall\, \varepsilon > 0$, $\exists\, N_2 \in \mathbb{N}$ s.t.

if $q > l \geq N_2$, then $|x_{l+1}| + |x_{l+2}| + \cdots + |x_q| < \varepsilon$

Therefore, for $N = \max\{N_1, N_2\}$,

if $n, q > N$,

$$\begin{cases} |S_n - x| < \varepsilon \text{ and} \\ |x_{N+1}| + |x_{N+2}| + \cdots + |x_q| < \varepsilon \end{cases} \quad \text{—} (*)$$

Let Σy_k be a rearrangement of Σx_n given by the bijection $f: \mathbb{N} \to \mathbb{N}$, i.e. $y_k = x_{f(k)}$, $\forall k \in \mathbb{N}$.

Let $M = \max\{f^{-1}(1), \cdots, f^{-1}(N)\}$,

then all the terms $x_1, \cdots, x_N$ are contained in $\{y_1, \cdots, y_M\}$.

$\therefore$ If $t_m = \sum_{k=1}^{m} y_k$, then $\forall m \geq M$, $(\& \ n > N)$

$$t_m - s_n = (y_1 + \cdots + y_M + \cdots + y_m) - (x_1 + \cdots + x_N + \cdots + x_n)$$

$$= \underbrace{(y_1 + \cdots + y_M) - (x_1 + \cdots + x_N)}_{(\text{no } x_1, \cdots, x_N \text{ remain})} + (y_{M+1} + \cdots + y_m) - (x_{N+1} + \cdots + x_n)$$

(no $x_1, \cdots, x_N$ in these terms)

is a sum of finite number of terms x_k with $k > N$.

$\Rightarrow \quad |t_m - s_n| \leq \sum_{k=N+1}^{q} |x_k| \quad$ for some q

By $(*)$, $\quad |t_m - s_n| < \varepsilon$.

Hence, $\forall \varepsilon > 0$, $\exists M > 0$ such that

if $m \geq M$, $\quad |t_m - x| \leq |t_m - s_n| + |s_n - x| < \varepsilon + \varepsilon = 2\varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\lim_{m \to \infty} t_m = x$

$\therefore \quad \Sigma y_k \to x = \Sigma x_n$.

$\#$

§9.2 Tests for Absolute Convergence

Thm 9.2.1 (Limit Comparison Test II)

Suppose $\begin{cases} \bullet\ x_n, y_n \neq 0, \quad \forall n = 1, 2, \cdots \\ \bullet\ \lim_{n\to\infty} \left|\frac{x_n}{y_n}\right| = r \text{ exists} \end{cases}$

Then (a) If $r \neq 0$, then

$$\sum x_n \text{ \underline{absolutely} convergent} \Leftrightarrow \sum y_n \text{ \underline{absolutely} convergent}$$

(b) If $\underline{r = 0}$ and $\sum y_n$ <u>absolutely</u> convergent,

then $\sum x_n$ is <u>absolutely</u> convergent (only $\sum y_n \Rightarrow \sum x_n$, $\not\Leftarrow$ in this case)

Pf: Recall <u>Limit Comparison Test (Thm 3.7.8)</u> that

if $x_n, y_n \underline{> 0}$, $r = \lim_{n\to\infty} \frac{x_n}{y_n}$ exists

then $\begin{cases} \bullet \text{ If } r \neq 0, \ \sum x_n \text{ converges} \Leftrightarrow \sum y_n \text{ converges} \\ \bullet \text{ If } r = 0, \ \sum y_n \text{ converges} \Rightarrow \sum x_n \text{ converges.} \end{cases}$

Applying Thm 3.7.8 to $\sum |x_n|$ & $\sum |y_n|$ ※

Recall also <u>Comparison Test (Thm 3.7.7)</u>: $0 \le x_n < y_n, \ \forall n \ge K$ (for some $K \in \mathbb{N}$)

then $\begin{cases} \text{(a) } \sum y_n \text{ converges} \Rightarrow \sum x_n \text{ converges} \\ \text{(b) } \sum x_n \text{ diverges} \Rightarrow \sum y_n \text{ diverges.} \end{cases}$

Thm 9.2.2 (**Root Test**) (Cauchy)

(a) If $\exists$ $r<1$ and $K \in \mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \leq r, \ \forall n \geq K,$$

then $\sum x_n$ is absolutely convergent.

(b) If $\exists K \in \mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \geq 1, \ \forall n \geq K,$$

then $\sum x_n$ is divergent.

Pf: (a) If $|x_n|^{\frac{1}{n}} \leq r$, $\forall n \geq K$

then $|x_n| \leq r^n$, $\forall n \geq K$

Since $\sum r^n$ is convergent for $0 \leq r < 1$,

Comparison Test 3.7.7 $\Rightarrow$ $\sum |x_n|$ is convergent.

(b) If $|x_n|^{\frac{1}{n}} \geq 1$, then $|x_n| \geq 1$, $\forall n \geq K$

$\Rightarrow$ $x_n \not\to 0$ as $n \to \infty$

$\Rightarrow$ $\sum x_n$ is divergent (n^{th} Term Test 3.7.3) #

Cor 9.2.3 Suppose $r = \lim_{n\to\infty} |x_n|^{\frac{1}{n}}$ exists.

Then
- $r<1 \Rightarrow \sum x_n$ is absolutely convergent
- $r>1 \Rightarrow \sum x_n$ is divergent.

(No conclusion for $r=1$. see Eg 9.2.7 (b) later)

Pf: If $r<1$, then $\forall\, r<r_1<1$, $\exists\, K\in\mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \le r_1 < 1, \quad \forall\, n\ge K,$$

then part (a) of Root Test $\Rightarrow$ $\sum x_n$ absolutely convergent.

If $r>1$, then $\exists\, K\in\mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} > 1, \quad \forall\, n\ge K,$$

then part (b) of Root Test $\Rightarrow$ $\sum x_n$ divergent. #

Thm 9.2.4 (Ratio Test) (D'Alembert)

Let $x_n \neq 0$, $\forall\, n=1,2,3,\cdots$

(a) If $\exists\, 0<r<1$ and $K\in\mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \le r, \quad \forall\, n\ge K,$$

then $\sum x_n$ is absolutely convergent

(b) If $\exists\, K\in\mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \ge 1, \quad \forall\, n\ge K,$$

then $\sum x_n$ is divergent.

Pf: (a) $\forall\, n\ge K$, $|x_n| \le r|x_{n-1}| \le r^2|x_{n-2}| \le \cdots \le r^{n-K}|x_K|$

If $0<r<1$, then $\sum y_n = \sum r^{n-K}|x_K| = \frac{|x_K|}{r^K}\sum r^n$ is convergent

Comparison Test 3.7.7 $\Rightarrow$ $\sum |x_n|$ is convergent.

i.e. $\sum x_n$ is absolutely convergent.

(b) $\forall n \geq K$, $|x_n| \geq |x_{n-1}| \geq |x_{n-2}| \geq \cdots \geq |x_K|$

$\therefore$ $x_n \not\to 0$ as $n \to \infty$ $\Rightarrow$ $\sum x_n$ is divergent. #

<u>Cor 9.2.5</u> If
- $x_n \neq 0$, $\forall n = 1, 2, 3, \cdots$, and
- $r = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$ exists

Then
- $r < 1 \Rightarrow \sum x_n$ is <u>absolutely convergent</u>.
- $r > 1 \Rightarrow \sum x_n$ is <u>divergent</u>

(No conclusion for $r = 1$. see Eg 9.2.7(c) later)

Pf: If $r < 1$, then $\forall r_1 \in (r, 1)$, $\exists K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| < r_1 < 1, \quad \forall n \geq K$$

Part (a) of Thm 9.2.4 $\Rightarrow$ $\sum x_n$ is absolutely convergent.

If $r > 1$, then $\exists K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| > 1, \quad \forall n \geq K$$

Part (b) of Thm 9.2.4 $\Rightarrow$ $\sum x_n$ is divergent. #

The Integral Test

Def (Improper Integral)

For $a \in \mathbb{R}$, if $\begin{cases} \bullet \ f \in R[a,b], \ \forall b > a, \text{ and} \\ \bullet \ \lim_{b \to +\infty} \int_a^b f \text{ exists (and } < +\infty.) \end{cases}$

then the improper integral $\int_a^\infty f$ is defined to be

$$\int_a^\infty f = \lim_{b \to +\infty} \int_a^b f .$$

Thm 9.2.6 (Integral Test)

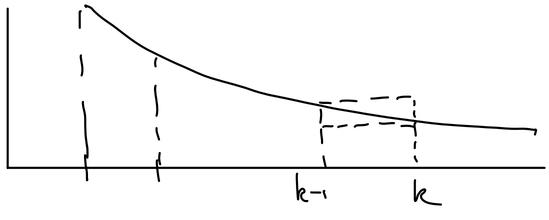
Let $f(t) > 0$, decreasing on $\{t \geq 1\}$.

Then $\sum_{k=1}^{\infty} f(k)$ converges $\Leftrightarrow$ $\int_1^\infty f = \lim_{b \to +\infty} \int_1^b f$ exists.

In this case,

$$\int_{n+1}^{\infty} f(t)dt \leq \sum_{k=1}^{\infty} f(k) - \sum_{k=1}^{n} f(k) \leq \int_n^\infty f(t)dt , \quad \forall n = 1, 2, \cdots$$

Pf:



$f > 0$ & decreasing $\Rightarrow$ $\forall k = 2, 3, \cdots$

$$f(k) \leq \int_{k-1}^{k} f(t)dt \leq f(k-1) \qquad \text{— } (*)_1$$

$$\Rightarrow \quad \sum_{k=2}^{n} f(k) \le \sum_{k=2}^{n} \int_{k-1}^{k} f(t)\,dt \le \underset{\substack{\parallel \\ f(1)+\cdots+f(n-1)}}{\sum_{k=2}^{n} f(k-1)}$$

Let $S_n = \sum_{k=1}^{n} f(k)$

Then, we have

$$S_n - f(1) \le \int_1^n f(t)\,dt \le S_{n-1}.$$

$\therefore \quad \lim_{n\to\infty} S_n$ exists $\Leftrightarrow \lim_{n\to\infty} \int_1^n f(t)\,dt$ exists (bdd, increasing)

& $\sum_{k=1}^{\infty} f(k)$ converges $\Leftrightarrow \int_1^\infty f$ exist. ($\lim_{n\to\infty}\int_1^n f$ exists $\Rightarrow \lim_{b\to\infty}\int_1^b f$ exists) will be proved next time

Using $(*)_1$ again, if $m>n$, then

$$\sum_{k=n+1}^{m} f(k) \le \sum_{k=n+1}^{m} \int_{k-1}^{k} f(t)\,dt \le \sum_{k=n+1}^{m} f(k-1)$$

$$\Rightarrow \quad S_m - S_n \le \int_n^m f(t)\,dt \le S_{m-1} - S_{n-1}$$

Hence, $\forall\, m>n$, we have

$$\int_{n+1}^{m+1} f(t)\,dt \le S_m - S_n \le \int_n^m f(t)\,dt$$

Letting $m\to\infty$, we have

$$\int_{n+1}^{\infty} f(t)\,dt \le S - S_n \le \int_n^\infty f(t)\,dt$$

where $S = \sum_{k=1}^{\infty} f(k)$. ※