§8.4 The Trigonometric Functions

Thm 8.4.1 $\exists$ functions $C:\mathbb{R}\to\mathbb{R}$ and $S:\mathbb{R}\to\mathbb{R}$ such that

(i) $C''(x) = -C(x)$ and $S''(x) = -S(x)$, $\forall x\in\mathbb{R}$.

(ii) $\begin{cases} C(0)=1 \\ C'(0)=0 \end{cases}$ and $\begin{cases} S(0)=0 \\ S'(0)=1 \end{cases}$

Pf: Define $C_n(x)$ and $S_n(x)$ inductively by

$$\begin{cases} C_1(x)=1 \\ S_1(x)=x \end{cases}$$

$$\begin{cases} S_n(x)=\int_0^x C_n(t)\,dt \\ C_{n+1}(x) = 1-\int_0^x S_n(t)\,dt \end{cases}$$

(i.e. starting with
$$\begin{tikzcd} C_1 \arrow[d] & C_2 \arrow[d] & C_3 \arrow[d] & & C_n \arrow[d] & C_{n+1} \arrow[d] & \cdots \\ S_1 \arrow[ur] & S_2 \arrow[ur] & S_3 \arrow[ur] & \cdots & S_n \arrow[ur] & \cdots \end{tikzcd}$$
)

Then "Induction": C_n & S_n are continuous, $\forall n$

$\Rightarrow$ integrable on any <u>bounded</u> interval

$\therefore$ All C_n & S_n are well-defined.

Moreover, by Fundamental Thm 7.3.5,

$$S_n'(x) = C_n(x) \quad \& \quad C_{n+1}'(x) = -S_n(x), \quad \forall x\in\mathbb{R},\ \forall n$$

Claim:
$$\begin{cases} C_{n+1}(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^{2n}}{(2n)!} \\[2ex] S_{n+1}(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} \end{cases}$$

Pf: (Ex! By induction)

Let $A > 0$.

If $x \in [-A, A]$ (i.e. $|x| \leq A$) and $m > n > 2A$, $\left(\frac{A}{2n}, \frac{A}{2m} < \frac{1}{4} \right)$

then

$$|C_m(x) - C_n(x)| = \left| (-1)^n \frac{x^{2n}}{(2n)!} + \cdots + (-1)^{m-1} \frac{x^{2(m-1)}}{(2(m-1))!} \right|$$

$$\leq \frac{A^{2n}}{(2n)!} + \cdots + \frac{A^{2m-2}}{(2m-2)!}$$

$$= \frac{A^{2n}}{(2n)!} \left[1 + \frac{(2n)!}{(2(n+1))!} A^2 + \frac{(2n)!}{(2(n+2))!} A^4 + \cdots + \frac{(2n)!}{(2(m-1))!} A^{2(m-1-n)} \right]$$

$$\leq \frac{A^{2n}}{(2n)!} \left[1 + \frac{A^2}{(2n)^2} + \frac{A^4}{(2n)^4} + \cdots + \frac{A^{2(m-1-n)}}{(2n)^{2(m-1-n)}} \right]$$

$$\leq \frac{A^{2n}}{(2n)!} \left[1 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^4 + \cdots + \left(\frac{1}{4}\right)^{2(m-1-n)} \right]$$

$$< \frac{16}{15} \cdot \frac{A^{2n}}{(2n)!}$$

Since $\lim_{n\to\infty} \frac{A^{2n}}{(2n)!} = 0$, Cauchy Criterion for Uniform Convergence implies C_n converges uniformly on $[-A, A]$, $\forall A > 0$

And hence, $C_n(x)$ converges $\forall x \in \mathbb{R}$.

Let $C(x) = \lim_{n\to\infty} C_n(x)$.

Then C_n converges uniformly to C on $[-A, A]$, $\forall A > 0$.

Hence Thm 8.2.2 $\Rightarrow$

C is cts on $[-A, A]$, $\forall A > 0$

and therefore, C is cts on $\mathbb{R}$.

Moreover, $C_n(0) = 1, \forall n \Rightarrow C(0) = 1$.

Since $S_n(x) = \int_0^x C_n(t)\, dt$

$$S_m(x) - S_n(x) = \int_0^x (C_m(t) - C_n(t))\, dt$$

$$\Rightarrow \quad |S_m(x) - S_n(x)| \leq \int_0^x |C_m(t) - C_n(t)|\, dt \quad \text{if } x \geq 0$$

(Cor 7.3.15) $\quad \left(\int_x^0 |C_m(t) - C_n(t)|\, dt, \text{ if } x < 0 \right)$

Then for $x \in [-A, A]$ & $m > n > 2A$,

$$|S_m(x) - S_n(x)| \leq \int_0^x \frac{16}{15} \cdot \frac{A^{2n}}{(2n)!}\, dt$$

$$\leq \frac{16}{15} \cdot \frac{A^{2n}}{(2n)!} \cdot A \quad \left(\text{similarly for } \int_x^0 \cdots \right)$$

$$\to 0 \text{ as } n \to \infty$$

$\therefore$ S_n converges uniformly on $[-A, A]$, $\forall A > 0$.

$\Rightarrow$ $S_n(x)$ converges $\forall x \in \mathbb{R}$.

Let $S(x) = \lim_{n\to\infty} S_n(x)$, $\forall x \in \mathbb{R}$

Then S_n converges uniformly to S on $[-A, A]$, $\forall A > 0$.

By Thm 8.2.2, S is cts on $\mathbb{R}$ (as S_n cts on $\mathbb{R}$, $\forall n$)

Since $S_n(0) = 0$, $\forall n$, we have $S(0) = 0$.

Now by Fundamental Thm of Calculus,

$$C_n'(x) = -S_{n-1}(x) \underset{(\text{uniform})}{\rightrightarrows} -S(x) \quad \text{on } [-A, A], \ \forall A > 0$$

Thm 8.2.3 $\Rightarrow$

$$C(x) = \lim_{n\to\infty} C_n(x) \quad \text{is differentiable and}$$

$$C'(x) = -S(x) \quad \text{on } [-A, A], \ \forall A > 0$$

Hence C is differentiable $\forall x \in \mathbb{R}$ and

$$C'(x) = -S(x), \ \forall x \in \mathbb{R}.$$

In particular, $C'(0) = -S(0) = 0$

Similarly, Fundamental Thm

$$\Rightarrow S_n'(x) = C_n(x) \rightrightarrows C(x) \quad \text{on } [-A, A], \ \forall A > 0$$

$\vdots$ (Ex!)

$\Rightarrow$ S is differentiable $\forall x \in \mathbb{R}$ &

$$S'(x) = C(x), \ \forall x \in \mathbb{R}$$

In particular, $S'(0) = C(0) = 1$.

Finally, combining the 2 formulae of 1st derivatives, we have

$$C''(x) = -S'(x) = -C(x) \quad \&$$

$$S''(x) = C'(x) = -S(x) \ .$$

$\divideontimes$