Thm 8.2.6 (Dini's Theorem)

Let
- $f_n : [a,b] \to \mathbb{R}$ be a monotone seq. of continuous functions
- $f_n \to f$ on $[a,b]$ (pointwise convergence)
- f is continuous

Then $f_n \rightrightarrows f$ on $[a,b]$ (uniform convergence)

Remark:

monotone $\begin{cases} \text{increasing seq.: } n \leq m \Rightarrow f_n(x) \leq f_m(x), \ \forall x \in [a,b] \\ \text{decreasing seq.: } n \leq m \Rightarrow f_n(x) \geq f_m(x), \forall x \in [a,b] \end{cases}$

Pf We assume f_n is a decreasing seq. The proof is similar for increasing sequence.

Let $g_n = f_n - f$.

Then g_n is decreasing, continuous and

$g_n \to 0$ (pointwise)

(Different proof from the Textbook)

Assume on the contrary that $g_n \not\rightrightarrows 0$ not uniform.

Then by Lemma 8.1.5,

$\exists \varepsilon_0 > 0$, a subseq g_{n_k} of g_n, and a seq $x_k \in [a,b]$

s.t. $$|g_{n_k}(x_k) - 0| \geq \varepsilon_0$$

$$\Rightarrow \quad g_{n_k}(x_k) \geq \varepsilon_0 \quad (\text{as } g_n \text{ decreasing} \Rightarrow g_n \geq 0)$$

Since $x_k \in [a,b]$, (x_k) is a bounded seq.

Then <u>Bolzano-Weierstrass Thm (Thm 3.4.8)</u> implies that

x_k has a convergence subseq $(x_{k_\ell})_{\ell=1}^{\infty}$

Let $\lim_{\ell\to\infty} x_{k_\ell} = z$.

Since $[a,b]$ is a closed interval, $z \in [a,b]$.

By assumption $g_n(z) \to 0$ as $n \to \infty$.

$\therefore$ $g_{n_{k_\ell}}(z) \to 0$ as $\ell \to \infty$.

$\Rightarrow$ $\exists\, L > 0$ s.t.

if $\ell \geq L$, then $0 \leq g_{n_{k_\ell}}(z) < \frac{\varepsilon_0}{2}$

In particular $0 \leq g_{n_{k_L}}(z) < \frac{\varepsilon_0}{2}$

For clarity of presentation, denote n_{k_L} by N.

Then $0 \leq g_N(z) < \frac{\varepsilon_0}{2}$.

Now using continuity of g_N $(= g_{n_{k_L}})$

$$\lim_{\ell\to\infty} g_N(x_{k_\ell}) = g_N(z) \qquad (\text{since } \lim_{\ell\to\infty} x_{k_\ell} = z)$$

$\Rightarrow$ $\exists\, L_1 > 0$ s.t. if $\ell \geq L_1$, then

$$g_N(x_{k_\ell}) < \frac{\varepsilon_0}{2}$$

Using the assumption that g_n is decreasing, we have

$$g_n(x_{k_\ell}) \le g_N(x_{k_\ell}) < \frac{\varepsilon_0}{2}, \quad \forall n \ge N = n_{k_L}$$

In particular, for $n = n_{k_\ell}$ with $\ell \ge \max\{L, L_1\}$, we have

$$\varepsilon_0 \le g_{n_{k_\ell}}(x_{k_\ell}) \le \frac{\varepsilon_0}{2}$$

which is a contradiction.

Therefore $g_n \rightrightarrows 0$ (uniform convergence) ※

<u>Remark</u>: The approach in Textbook requires the fact that for any given function $t \mapsto \delta(t) > 0$ on $[a,b]$ $\exists$ finitely many $t_i \in [a,b]$, $i = 1, \cdots, \ell$ such that

$$[a,b] \subset \bigcup_{i=1}^{\ell} (t_i - \delta(t_i), t_i + \delta(t_i)).$$

This needs the Thm 5.5.5 which is not covered in MATH2050.

These two proofs use different versions of the fact that $[a,b]$, a closed & bounded interval, is <u>compact</u>:

(i) Any sequence in $[a,b]$ has a subsequence converges to some point in $[a,b]$.

(ii) For any open cover of $[a,b]$, $[a,b] \subset \bigcup_{\lambda} (\alpha_\lambda, \beta_\lambda)$, (where $(\alpha_\lambda, \beta_\lambda)$ are open intervals, could be infinitely many) has finite subcover, i.e.

$\exists$ finitely many λ_i, $i=1,\cdots,\ell$ such that

$$[a,b] \subset \bigcup_{i=1}^{\ell} (\alpha_{\lambda_i}, \beta_{\lambda_i})$$

Detail discussion and proof are skipped.

§ 8.3 The Exponential and Logarithmic Functions

The Exponential Function

Thm 8.3.1 $\exists$ a function $E : \mathbb{R} \to \mathbb{R}$ s.t.

(i) $E'(x) = E(x)$, $\forall x \in \mathbb{R}$

(ii) $E(0) = 1$

Pf: Let $E_1(x) = 1 + x$

$$E_2(x) = 1 + \int_0^x E_1 = 1 + \int_0^x (1+t)\,dt = 1 + x + \frac{x^2}{2}$$

$$\vdots$$

$$E_{n+1}(x) = 1 + \int_0^x E_n \quad , \quad \forall n = 1, 2, 3 \cdots$$

Then "Induction" implies for all $n = 1, 2, 3, \cdots$

$$E_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \qquad (\text{Ex!})$$

Consider a closed interval $[-A, A]$. $(A > 0)$

Then for $x \in [-A, A]$ and $m > n > 2A$, we have

$$\left| E_m(x) - E_n(x) \right| = \left| \frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!} \right|$$

$$\leq \frac{A^{n+1}}{(n+1)!} + \cdots + \frac{A^m}{m!} \qquad (\text{since } |x| \leq A)$$

$$\leq \frac{A^{n+1}}{(n+1)!} \left(1 + \frac{A}{n+2} + \cdots + \frac{A^{m-n-1}}{m(m-1)\cdots(n+2)} \right)$$

$$\leq \frac{A^{n+1}}{(n+1)!}\left(1+\frac{A}{n}+\cdots+\frac{A^{m-n-1}}{n^{m-n-1}}\right)$$

$$\leq \frac{A^{n+1}}{(n+1)!}\left[1+\frac{1}{2}+\cdots+\left(\frac{1}{2}\right)^{m-n-1}\right] \quad (\text{since } n>2A)$$

$$\leq \frac{2A^{n+1}}{(n+1)!}$$

Taking sup. over $[-A,A]$, we have $\forall\, m>n>2A$

$$\|E_m-E_n\|_{[-A,A]} \leq \frac{2A^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as } n\to\infty$$

Cauchy Criterion for Uniform Convergence (Thm 8.1.10) implies

$E_n(x)$ converges uniformly to some function on $[-A,A]$

Since $A>0$ is arbitrary, we conclude that

$E_n(x)$ converges for all $x\in\mathbb{R}$ (not necessary uniform on $\mathbb{R}$)

It is because, $\forall\, x\in\mathbb{R}$, we can find an $A>0$ s.t.

$x\in[-A,A]$. Then the uniform convergence on $[-A,A]$

implies $E_n(x)$ converges.

Denote the (pointwise) limit by

$$E(x) \xlongequal{\text{denote}} \lim_{n\to\infty} E_n(x), \quad \forall x\in\mathbb{R}.$$

Note that $\quad E_n(x) = 1+\int_0^x E_{n-1}$

$$\Rightarrow \quad E_n(0)=1, \quad \forall\, n=2,3,\cdots \quad (E_1(0)=1 \text{ is clear})$$

Hence $\quad E(0) = \lim_{n\to\infty} E_n(0) = 1$.

Also by Fundamental Thm of Calculus (2nd Form) Thm 7.3.5

and $\quad E_n(x) = 1 + \int_0^x E_{n-1}$,

we have $\quad E_n'(x) = E_{n-1}(x)$

$\therefore \ \forall A > 0$,

$$\left(E_n\big|_{[-A,A]}\right)' = E_{n-1}\big|_{[-A,A]} \rightrightarrows E\big|_{[-A,A]} \text{ (uniform)}$$

Then by Thm 8.2.3, together with $E_{n+1}\big|_{[-A,A]}(0) \to E(0)$

we have $\quad E\big|_{[-A,A]}$ is differentiable and

$$\left(E\big|_{[-A,A]}\right)' = E\big|_{[-A,A]}$$

Since $A>0$ is arbitrary, this implies $E'(x)$ exists $\forall x \in \mathbb{R}$ and

$$E'(x) = E(x)$$

Cor 8.3.2 The function E has derivative of every order and

$$E^{(n)}(x) = E(x), \quad \forall x \in \mathbb{R}.$$

Pf: Easy, by induction.

Cor 8.3.3 If $x>0$, then $E(x)>1+x$

Pf: From $E_n(x)=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}$, we have

$$m>n \Rightarrow E_m(x)>E_n(x), \quad \forall x>0$$

Letting $m\to\infty$, and take $n>1$, we have

$$E(x)\geq E_n(x)>E_1(x)=1+x, \quad \forall x>0 \quad ※$$

Thm 8.3.4: $E:\mathbb{R}\to\mathbb{R}$ is the unique function satisfying

$$(*)\begin{cases} E'(x)=E(x), & \forall x\in\mathbb{R} \\ E(0)=1 \end{cases}$$

Pf: Suppose that E_1 & E_2 satisfy $(*)$.

Let $F=E_1-E_2$.

Then F is differentiable and

$$\begin{cases} F'=E_1'-E_2'=E_1-E_2=F \\ F(0)=E_1(0)-E_2(0)=0 \end{cases}$$

Moreover, induction $\Rightarrow$ F has derivatives of every order

and $\qquad F^{(n)}=F, \quad \forall n=1,2,3,\cdots$

Hence $\qquad F^{(n)}(0)=F(0)=0, \quad \forall n=1,2,3,\cdots$

Applying Taylor's Thm 6.4.1 to $F|_{[0,x]}$ for $x>0$

or $F|_{[x,0]}$ for $x<0$,

we have for $x>0$

$$F(x) = F(0) + F'(0)\,x + \cdots + \frac{F^{(n-1)}(0)}{(n-1)!}x + \frac{F^{(n)}(c_n)}{n!}x^n$$

$$= \frac{F(c_n)}{n!}x^n \qquad \text{for some } c_n \in [0,x].$$

Since F is cts on $[0,x]$, F is bdd on $[0,x]$.

$\therefore$ $\exists\, K>0$ (depends on x) such that

$$|F(c_n)| \le K \qquad (\forall n = 1,2,\cdots)$$

$$\Rightarrow \quad |F(x)| \le K\frac{x^n}{n!}$$

Since $\lim_{n\to\infty}\frac{x^n}{n!} = 0$, letting $n\to\infty$, we have $|F(x)| = 0$.

$\therefore$ $F(x)\equiv 0$, $\forall x>0$

Similarly for $x<0$, we also have $F(x)\equiv 0$, $\forall x<0$.

All together $F(x)\equiv 0$.

i.e. $E_1(x) \equiv E_2(x)$

$\therefore$ The function E is unique.

Def 8.3.5 The Unique function $E: \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} E'(x) = E(x), \ \forall x \in \mathbb{R} & \text{—— (i)} \\ E(0) = 1 & \text{—— (ii)} \end{cases}$$

is called the <u>exponential function</u> and is denoted by

$$e^x \quad \text{or} \quad \exp(x)$$

The number $e = E(1)$ is called the <u>Euler's number</u>.

Thm 8.3.6 Exponential function E satisfies

- $E(x) \neq 0, \ \forall x \in \mathbb{R}$ —— (iii)
- $E(x+y) = E(x)E(y) \ \forall x, y \in \mathbb{R}$ —— (iv)
- $E(r) = e^r, \ \forall r \in \mathbb{Q}$ —— (v)

<u>Remarks</u>:
- (iv) justifies the use of notation $e^x = E(x)$:

$$e^{x+y} = e^x e^y, \quad \forall x, y \in \mathbb{R}$$

- In (v), "RHS" means the <u>rational power</u> of the number e

<u>Pf</u>: (iii) Suppose on the contrary that $E(\alpha) = 0$ for some $\alpha \in \mathbb{R}$.

Since $E(0) = 1$, $\alpha \neq 0$.

Let J_α = closed interval $[0, \alpha]$ or $[\alpha, 0]$ depends on the sign of α.

and $K>0$ such that $|E(x)| \le K$, $\forall x \in J_\alpha$.

As E has derivative of all order, Taylor's Thm 6.4.1

(base at $x_0 = \alpha$) implies $\forall n = 1, 2, 3, \cdots$

$$E(0) = E(\alpha) + \frac{E'(\alpha)}{1!}(0-\alpha) + \cdots + \frac{E^{(n-1)}(\alpha)}{(n-1)!}(0-\alpha)^{n-1}$$

$$+ \frac{E^{(n)}(c_n)}{n!}(0-\alpha)^n \qquad \text{for some } c_n \in J_\alpha.$$

$$\Rightarrow \quad 1 = E(\alpha) + E(\alpha)(-\alpha) + \frac{E(\alpha)}{2!}(-\alpha)^2 + \cdots + \frac{E(\alpha)}{(n-1)!}(-\alpha)^{n-1}$$

$$+ \frac{E(c_n)}{n!}(-\alpha)^n$$

since $E(0) = 1$, and $E^{(k)} = E \quad \forall k = 1, 2, \cdots$

By $E(\alpha) = 0$, $\quad 1 = \dfrac{E(c_n)}{n!}(-\alpha)^n$

$$\Rightarrow \quad 1 \le \frac{K|\alpha|^n}{n!}, \quad \forall n = 1, 2, \cdots$$

$$(\longrightarrow 0 \quad \text{as } n \to \infty)$$

which is impossible. $\therefore E(\alpha) \neq 0$, $\forall \alpha \in \mathbb{R}$.

Pf: (iv) Fix y and consider the ratio

$$G(x) = \frac{E(x+y)}{E(y)} \quad \text{as a function of } x.$$

$G(x)$ is well-defined since $E(y) \neq 0$ by (iii).

$$G(0) = \frac{E(y)}{E(y)} = 1.$$

E differentiable $\Rightarrow$ G differentiable and

$$G'(x) = \frac{E'(x+y)}{E(y)} \quad \text{(by Chain rule)}$$

$$= \frac{E(x+y)}{E(y)} = G(x) \qquad \text{(by (i))}$$

By Thm 8.3.4, $G(x) = E(x)$, $\forall x \in \mathbb{R}$

$\therefore \quad E(x+y) = E(x)E(y) \quad \forall x, y \in \mathbb{R}.$

Pf: (v) By (iv)

$$E(nx) = E((n-1)x + x) = E((n-1)x)E(x)$$

$$= \left(E((n-2)x)E(x)\right)E(x)$$

$$\vdots$$

$$= E(0)\,E(x)^n$$

$$= E(x)^n, \quad \forall n = 1, 2, 3 \cdots$$

Clearly, it also holds for $n=0$: $E(0 \cdot x) = (E(x))^0 = 1$, $\forall x$

Putting $x = \frac{1}{n}$, we have

$$e = E(1) = E(n \cdot \tfrac{1}{n}) = [E(\tfrac{1}{n})]^n$$

$\therefore \quad E(\frac{1}{n}) = e^{\frac{1}{n}}$ as n-root of the number e.

For $m \in \mathbb{Z}$,

Case (1) $\underline{m \geq 0}$

Then $E\left(\frac{m}{n}\right) = \left(E\left(\frac{1}{n}\right)\right)^m = \left(e^{\frac{1}{n}}\right)^m = e^{\frac{m}{n}}$.

Case (2), $\underline{m < 0}$.

Then $-m > 0$ and

$$1 = E(0) = E\left(\frac{m}{n} + \frac{(-m)}{n}\right) = E\left(\frac{m}{n}\right) E\left(\frac{(-m)}{n}\right)$$

$$\therefore \quad E\left(\frac{m}{n}\right) = \frac{1}{E\left(\frac{(-m)}{n}\right)} = \frac{1}{e^{\frac{(-m)}{n}}} \qquad \text{since } -m > 0$$

$$= e^{\frac{m}{n}}$$

※

<u>Thm 8.3.7</u>
- Exponential function E is <u>strictly increasing</u> on $\mathbb{R}$ and
- $E(\mathbb{R}) = \{y \in \mathbb{R} : y > 0\}$.

Further,

$$\left.\begin{array}{l} \bullet \ \lim\limits_{x \to -\infty} E(x) = 0 \\ \bullet \ \lim\limits_{x \to +\infty} E(x) = +\infty \end{array}\right\} \text{—— (vi)}$$

Pf: E differentiable on $\mathbb{R}$

$\Rightarrow$ E continuous on $\mathbb{R}$.

It's proved in (iii) in Thm 8.3.6 that $E(x) \neq 0, \forall x \in \mathbb{R}$.

$\therefore$ $E(0) = 1$ $\Rightarrow$ $E(x) > 0$, $\forall x \in \mathbb{R}$

Otherwise, intermediate value thm $\Rightarrow E(x_0)=0$ for some x_0

which is a contradiction.

Hence $E'(x) = E(x) > 0 \quad \forall x \in \mathbb{R}$

which implies E is strictly increasing.

By Cor 8.3.3, $E(x) > 1+x \quad \forall x > 0$

$$\Rightarrow \lim_{x \to +\infty} E(x) = +\infty .$$

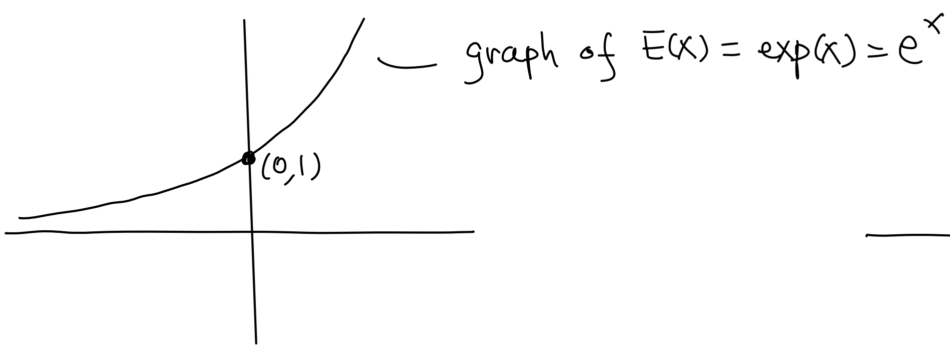
Using (iv), if $x<0$, then $E(x) = \dfrac{1}{E(|x|)}$

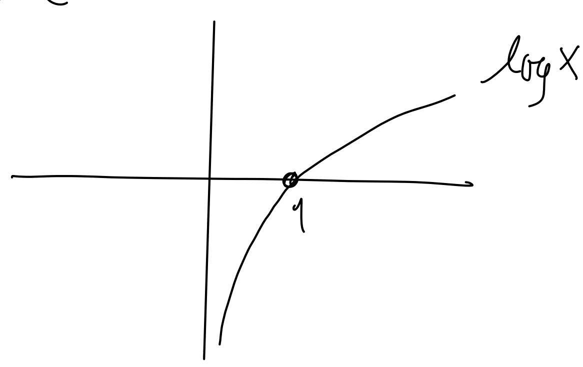
$$\therefore \lim_{x \to -\infty} E(x) = \lim_{|x| \to +\infty} \frac{1}{E(|x|)} = 0 .$$

Finally, with continuity of E and the values of the limits, intermediate value thm implies

$$\forall y>0, \ \exists x \in \mathbb{R} \text{ s.t. } y = E(x).$$

Therefore $E(\mathbb{R}) = \{ y \in \mathbb{R} : y > 0 \}$.





The Logarithm Function

> **Def 8.3.8** The *inverse function* of E is called the *logarithm* (or the *natural logarithm*).

Notation: In the Textbook, *logarithm* is denoted by "L".

Other common notations are "$\ln$" or "$\log$".

(used more in graduate textbook of research articles in mathematics)

Note: By definition

$$\begin{cases} (L \circ E)(x) = x, & \forall x \in \mathbb{R} \\ (E \circ L)(y) = y, & \forall y > 0 \end{cases} \qquad (E : \mathbb{R} \to \{y > 0\} = E(\mathbb{R}))$$

i.e. $\ln e^x = x$, $e^{\ln y} = y$

(or $\log e^x = x$, $e^{\log y} = y$)

Thm 8.3.9 • The logarithm $L:\{x>0\}\to\mathbb{R}$ is a strictly increasing function with domain $\{x\in\mathbb{R}: x>0\}$ and $L(\{x>0\})=\mathbb{R}$.

- $L'(x)=\frac{1}{x}$, $\forall x>0$ ——— (vii)
- $L(xy)=L(x)+L(y)$, $\forall x>0, y>0$ ——— (viii)
- $L(1)=0$ & $L(e)=1$ ——— (ix)
- $L(x^r)=rL(x)$, $\forall x>0$ and $r\in\mathbb{Q}$ ——— (x)
- $\lim_{x\to 0^+} L(x)=-\infty$ & $\lim_{x\to+\infty} L(x)=+\infty$ ——— (xi)

Pf: All are easy from the definition. (Ex!)

Note that in property (x), $L(x^r)=rL(x)$ actually works for irrational number α: $L(x^\alpha)=\alpha L(x)$.

Although we used it a lot, x^α is not yet definited in the Textbook for $\alpha\notin\mathbb{Q}$.

Power Functions

Def 8.3.10 If $\alpha\in\mathbb{R}$ and $x>0$, then

$$x^\alpha \overset{def}{=\!=} e^{\alpha\ln x}=E(\alpha L(x))=e^{\alpha\log x}$$

The function $x\mapsto x^\alpha$ for $x>0$ is called the power function with exponent α.

Note: If $\alpha = r \in \mathbb{Q}$, then for $x>0$

$$E(\alpha L(x)) = E(rL(x)) = E(L(x^r)) \quad (\text{by property } (x))$$
$$= x^r$$

$\therefore$ Def 8.3.10 is consistent with previous definition for $r \in \mathbb{Q}$.

Thm 8.3.11 If $\alpha \in \mathbb{R}$, $x, y \in (0,\infty)$, then

(a) $1^\alpha = 1$,

(b) $x^\alpha > 0$,

(c) $(xy)^\alpha = x^\alpha y^\alpha$,

(d) $\left(\frac{x}{y}\right)^\alpha = \frac{x^\alpha}{y^\alpha}$

Pf: (Easy Ex!)

Thm 8.3.12 If $\alpha, \beta \in \mathbb{R}$, $x \in (0,\infty)$, then

(a) $x^{\alpha+\beta} = x^\alpha x^\beta$,

(b) $(x^\alpha)^\beta = x^{\alpha\beta} = (x^\beta)^\alpha$,

(c) $x^{-\alpha} = \frac{1}{x^\alpha}$,

(d) If $\alpha < \beta$, then $x^\alpha < x^\beta$ for $x > 1$

Pf: (Easy Ex!)

Thm 8.3.13 For $\alpha \in \mathbb{R}$,

$x \mapsto x^{\alpha}$ is continuous and differentiable on $(0,\infty)$, and

$$D x^{\alpha} = \alpha x^{\alpha-1}$$

Pf: Chain rule $\Rightarrow$ x^{α} is differentiable & hence continuous

and
$$D x^{\alpha} = D\big(E(\alpha L(x))\big) = E'(\alpha L(x))\, D(\alpha L(x))$$
$$= E(\alpha L(x)) \cdot \alpha\, D(L(x))$$
$$= \alpha x^{\alpha} \cdot \frac{1}{x}$$
$$= \alpha x^{\alpha-1}.$$

The Function $\log_a$ (logarithm of x to the base a)

Def 8.3.14 Let $a>0$ and $a \neq 1$.

$$\log_a(x) \overset{\text{def}}{=\!=} \frac{\ln x}{\ln a} = \frac{\log x}{\log a} \quad \text{for } x>0.$$