(Cont'd from last time)

Let $c \in I$, then mean value thm $\Rightarrow$ for $x \in I$ & $x \neq c$

$$(f_m - f_n)(x) - (f_m - f_n)(c) = (f_m' - f_n')(z)\,(x-c)$$ for some z between x & c.

$$\therefore \left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| = |f_m'(z) - f_n'(z)|$$
$$\leq \|f_m' - f_n'\|_I$$

Hence $\forall \varepsilon > 0$,

$$\left| \frac{f_m(x) - f_m(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| < \frac{\varepsilon}{2(b-a)} \quad \text{for } m, n \geq H_1$$

Letting $m \to \infty$ and using $f_m \rightrightarrows f$, we have for $x \neq c$

$$\left| \frac{f(x) - f(c)}{x-c} - \frac{f_n(x) - f_n(c)}{x-c} \right| \leq \frac{\varepsilon}{2(b-a)} \quad \text{for } n \geq H_1$$

Now using $f_n' \rightrightarrows g$ again

for the same $\varepsilon > 0$, $\exists\, N = N(\varepsilon) \in \mathbb{N}$ s.t.

$$|f_n'(c) - g(c)| < \varepsilon \quad \text{for } n \geq N$$

Then let $K = \max\{H_1, N\} \in \mathbb{N}$

$$\left| \frac{f(x) - f(c)}{x-c} - g(c) \right| \leq \left| \frac{f(x) - f(c)}{x-c} - \frac{f_K(x) - f_K(c)}{x-c} \right|$$
$$+ \left| \frac{f_K(x) - f_K(c)}{x-c} - f_K'(c) \right| + |f_K'(c) - g(c)|$$

$$< \left(1+\frac{1}{2(b-a)}\right)\varepsilon + \left|\frac{f_K(x)-f_K(c)}{x-c} - f_K'(c)\right|$$

Note that for the same $\varepsilon>0$, $\exists\ \delta_{\varepsilon,c}>0$ such that

$$\left|\frac{f_K(x)-f_K(c)}{x-c} - f_K'(c)\right| < \varepsilon, \quad \text{if } |x-c|<\delta_{\varepsilon,c}\ (x\neq c).$$

Therefore, we have proved that $\forall \varepsilon>0$, $\exists\ \delta_{\varepsilon,c}>0$ s.t.

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| < \left(2+\frac{1}{2(b-a)}\right)\varepsilon \text{ provided } |x-c|<\delta_{\varepsilon,c}.$$

Since $\varepsilon>0$ is arbitrary,

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \text{ exists \& equals } g(c).$$

As $c\in I$ is arbitrary, f is differentiable on I and

$$f' = g. \quad ※$$

<u>Interchange of Limit and Integral</u>

<u>Thm 8.2.4</u> Let
- $f_n \in R[a,b]$ for $n=1,2,3,\cdots$ (Riemann integrable)
- $f_n \rightrightarrows f$ on $[a,b]$ (converges uniformly on $[a,b]$ to f)

Then $f\in R[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$$

(i.e. f_n converges <u>uniformly</u> $\Rightarrow$ $\lim_{n\to\infty}\int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n$)

Pf: By Cauchy Criterion for Uniform Convergence (Thm 8.1.10),

$\forall \varepsilon > 0$, $\exists H(\varepsilon) > 0$ s.t.

if $m > n \geq H(\varepsilon)$, then $\|f_m - f_n\|_{[a,b]} < \varepsilon$

i.e. $-\varepsilon < f_m(x) - f_n(x) < \varepsilon \quad \forall x \in [a,b]$

Hence
$$-\varepsilon(b-a) \leq \int_a^b f_m - \int_a^b f_n \leq \varepsilon(b-a) \quad \text{———} (*)_1$$

i.e.
$$\left| \int_a^b f_m - \int_a^b f_n \right| \leq \varepsilon(b-a)$$

Since $\varepsilon > 0$ is arbitrary, this implies the seq. of numbers $\left(\int_a^b f_n\right)$ is a Cauchy sequence.

$\therefore \lim_{n\to\infty} \int_a^b f_n = A$ exists, (denoted by A).

$\Rightarrow \forall \varepsilon > 0$, $\exists K(\varepsilon) > 0$ s.t.

$$\left| \int_a^b f_n - A \right| < \varepsilon, \text{ for } n \geq K(\varepsilon). \quad \text{———} (*)_2$$

And letting $m \to \infty$ in the ineq. before $(*)_1$, we have

$\forall \varepsilon > 0$, $\exists H(\varepsilon) > 0$ s.t. if $n \geq H(\varepsilon)$, then

$$-\varepsilon \leq f(x) - f_n(x) \leq \varepsilon$$

i.e. $|f_n(x) - f(x)| \leq \varepsilon \quad \forall x \in [a,b]$ ——— $(*)_3$

Now, let $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{\ell}$ be a tagged partition of $[a,b]$.

If $n \geq \max\{H(\varepsilon), K(\varepsilon)\}$, we have

$$|S(f_n;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})| = \left|\sum_{i=1}^{l} f_n(t_i)(x_i - x_{i-1}) - \sum_{i=1}^{l} f(t_i)(x_i - x_{i-1})\right|$$

$$= \left|\sum_{i=1}^{l}(f_n(t_i) - f(t_i))(x_i - x_{i-1})\right|$$

$$\leq \sum_{i=1}^{l} |f_n(t_i) - f(t_i)|(x_i - x_{i-1})$$

$$\leq \varepsilon \sum_{i=1}^{l}(x_i - x_{i-1}) \qquad (\text{by } (*)_3)$$

$$= \varepsilon(b-a)$$

Then

$$|S(f;\dot{\mathcal{P}}) - A| \leq |S(f;\dot{\mathcal{P}}) - S(f_n;\dot{\mathcal{P}})| + |S(f_n,\dot{\mathcal{P}}) - A|$$

$$\leq \varepsilon(b-a) + \left|S(f_n;\dot{\mathcal{P}}) - \int_a^b f_n\right| + \left|\int_a^b f_n - A\right|.$$

$$< \varepsilon(b-a+1) + \left|S(f_n;\dot{\mathcal{P}}) - \int_a^b f_n\right|$$

Finally, fix an $n_0 \geq \max\{H(\varepsilon), K(\varepsilon)\}$ and using $f_{n_0} \in \mathcal{R}[a,b]$, $\exists\, \delta_{\varepsilon, n_0} > 0$ (depends on n_0 too) s.t.

if $\|\dot{\mathcal{P}}\| < \delta_{\varepsilon, n_0}$, then $\left|S(f_{n_0};\dot{\mathcal{P}}) - \int_a^b f_{n_0}\right| < \varepsilon$.

Hence $\forall \varepsilon > 0$, if $\|\dot{\mathcal{P}}\| < \delta_{\varepsilon, n_0}$, we have

$$|S(f;\dot{\mathcal{P}}) - A| < \varepsilon(b-a+1) + \varepsilon = \varepsilon(b-a+2).$$

Since $\varepsilon > 0$ is arbitrary, we have proved that

$$f \in R[a,b] \text{ and } \int_a^b f = A = \lim_{n\to\infty} \int_a^b f_n . \quad \#$$

Thm 8.2.5 ((Uniform) Bounded Convergence Theorem)

Let
- $f_n \in R[a,b]$ $\forall n = 1, 2, 3, \cdots$ (Riemann integrable)
- $f_n \to f$ on $[a,b]$ (pointwise convergence)
- $f \in R[a,b]$
- $\exists B > 0$ such that $\|f_n\|_{[a,b]} \le B$, $\forall n = 1, 2, 3 \cdots$

  (i.e. $|f_n(x)| \le B$, $\forall x \in [a,b]$ & $\forall n = 1, 2, 3, \cdots$)

Then $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n$

Pf: Omitted

Remark: The condition in Bounded Convergence Thm is weaker than

Thm 8.2.4