eg 8.1.2 (d) $\forall x \in \mathbb{R}$, $|F_n(x) - F(x)| \le \frac{1}{n} < \varepsilon$ $\left(\Rightarrow n > \frac{1}{\varepsilon}\right)$

Only need to choose $K(\varepsilon) = [\frac{1}{\varepsilon}] + 1$

which is <u>independent of x</u> and works for all $x \in \mathbb{R}$.

<u>Def 8.1.4</u> (<u>Uniform Convergence</u>)

A seq. $f_n : A \to \mathbb{R}$ <u>converges uniformly on A_0</u> $\subseteq A$ to a function

$$f : A_0 \to \mathbb{R}$$

if $\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ (depends on ε, but <u>not</u> on $x \in A_0$)

s.t. if $n \ge K(\varepsilon)$, then

$$|f_n(x) - f(x)| < \varepsilon, \ \forall x \in A_0.$$

<u>Remarks</u>: (i) In this case, we say that

(f_n) is <u>uniformly convergent on A_0</u>, and denoted by

- $f_n \rightrightarrows f$ on A_0 or
- $f_n(x) \rightrightarrows f(x)$ for $x \in A_0$

(Or in some other books, $f_n \to f$ uniformly on A_0)

(ii) uniform convergence $\Longrightarrow$ pointwise convergence

i.e. "$f_n \rightrightarrows f$ on A_0" $\Longrightarrow$ "$f_n \to f$ on A_0"

(Easy from the definitions)

Lemma 8.1.5: $f_n : A \to \mathbb{R}$ does not converge uniformly on $A_0 \subseteq A$ to $f : A_0 \to \mathbb{R}$

$\iff \exists$
- $\varepsilon_0 > 0$,
- a subsequence (f_{n_k}) of (f_n), and
- a seq. $x_k \in A_0$

such that $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$, $\forall\, k = 1, 2, 3, \cdots$

Pf: Negation of Def. 8.1.4:

$\exists\, \varepsilon_0 > 0$, such that $\forall\; k\,(= K(\varepsilon_0)) \in \mathbb{N}$,

the statement

"if $n \geq k$, then $|f_n(x) - f(x)| < \varepsilon_0$, $\forall\, x \in A_0$"

doesn't hold.

i.e. $\exists\, n_k\,(\geq k) \in \mathbb{N}$ s.t.

"$|f_{n_k}(x) - f(x)| < \varepsilon_0$, $\forall\, x \in A_0$" doesn't hold.

$\therefore \exists\, x_k \in A_0$ s.t. $|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$

All together, $\exists\, \varepsilon_0 > 0$, (f_{n_k}) subseq & $(x_k) \subset A_0$ s.t.

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 .$$

※

Eg 8.1.6

(a) eg 8.1.2 (a) $f_n(x) = \frac{x}{n}$, $f(x)=0$ $(A_0=\mathbb{R})$

Consider $n_k = k$, $x_k = k \in \mathbb{R}$. Then

$$|f_{n_k}(x_k) - f(x_k)| = \left|\frac{x_k}{n_k} - 0\right| = \left|\frac{k}{k}\right| = 1$$

$\therefore$ Choosing $\varepsilon_0 = 1$, then Lemma 8.1.5 $\Rightarrow$ $f_n \not\rightrightarrows f$ on $\mathbb{R}$

(b) eg 8.1.2 (b) $g_n(x) = x^n$, $g(x) = \begin{cases} 0, & |x|<1 \\ 1, & x=1 \end{cases}$, $A_0 = (-1,1]$

Consider $n_k = k$, $x_k = \left(\frac{1}{2}\right)^{1/k}$ $(|x_k|<1)$

Then $$|g_{n_k}(x_k) - g(x_k)| = \left|\left[\left(\frac{1}{2}\right)^{1/k}\right]^k - 0\right| = \frac{1}{2}, \quad \forall k$$

Choosing $\varepsilon_0 = 1/2$, Lemma 8.1.5 $\Rightarrow$ $x^n \not\rightrightarrows g$ on $(-1,1]$.

(c) eg 8.1.2 (c) $h_n(x) = \frac{x^2+nx}{n}$, $h(x) = x$, $A_0 = \mathbb{R}$

Consider $n_k = k$, $x_k = -k$,

Then $$|h_{n_k}(x_k) - h(x_k)| = \left|\frac{(x_k)^2 + n_k x_k}{n_k} - x_k\right|$$

$$= \left|\frac{(-k)^2 + k\cdot(-k)}{k} - (-k)\right|$$

$$= k \geq 1 \qquad (\to \infty)$$

Choosing $\varepsilon_0 = 1$, lemma 8.1.5 $\Rightarrow$ $h_n \not\rightrightarrows h$ on $\mathbb{R}$.

Def 8.1.7 (Uniform Norm) (supnorm in some other books)

If $\varphi : A \to \mathbb{R}$ is bounded on A (i.e. $\varphi(A)$ is a bounded subset of $\mathbb{R}$), then we define the uniform norm of φ on A by

$$\|\varphi\|_A = \sup\{|\varphi(x)| : x \in A\}.$$

Remark: $\|\varphi\|_A \leq \varepsilon \iff |\varphi(x)| \leq \varepsilon, \forall x \in A.$

Lemma 8.1.8: $f_n \rightrightarrows f$ on $A \iff \|f_n - f\|_A \to 0.$

Pf: ($\Rightarrow$) $f_n \rightrightarrows f$ on A.

By Def 8.1.4, $\forall \varepsilon > 0$, $\exists K(\frac{\varepsilon}{2}) \in \mathbb{N}$ s.t. if $n \geq K(\frac{\varepsilon}{2})$, then

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \forall x \in A$$

$\therefore \forall \varepsilon > 0$, $\exists N(\varepsilon) = K(\frac{\varepsilon}{2}) \in \mathbb{N}$ s.t. if $n \geq N(\varepsilon)$

$$\|f_n - f\|_A \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{(by remark above)}$$

i.e. $\|f_n - f\|_A \to 0$ as $n \to \infty$.

($\Leftarrow$) If $\|f_n - f\|_A \to 0$. Then $\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ s.t. if $n \geq K(\varepsilon)$, $\|f_n - f\|_A < \varepsilon$.

$$\Rightarrow |f_n(x) - f(x)| < \varepsilon, \forall x \in A.$$

$\therefore f_n \rightrightarrows f$ on A. ⨳

Eg 8.1.9

(a) eg 8.1.2(a), $f_n(x) = \frac{x}{n}$ on $\mathbb{R}$, $f(x) = 0$, on $\mathbb{R}$.

$f_n(x) - f(x) = \frac{x}{n}$ is unbounded, $\|f_n - f\|_{\mathbb{R}}$ is not defined.

However, if one consider only on the interval $A = [0,1]$.

Then $f_n(x) - f(x) = \frac{x}{n}$ is bounded on $[0,1]$,

and $\|f_n - f\|_{[0,1]} = \sup\{|\frac{x}{n}| : x \in [0,1]\}$

$= \frac{1}{n}$ ($\to 0$ as $n \to \infty$)

$\therefore$ $f_n|_{[0,1]} \rightrightarrows \underset{f}{\overset{\shortparallel}{0}}$ on $[0,1]$

(in fact $f_n \rightrightarrows f$ on any bounded subset, but $\not\rightrightarrows$ on unbounded subset)

(b) eg 8.1.2(b), consider only on $[0,1] \subseteq A_0$. $A_0 = (-1,1]$

Then $g_n(x) = x^n$, $g(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1. \end{cases}$

$\|g_n - g\|_{[0,1]} = \sup\{|g_n(x) - g(x)| : x \in [0,1]\}$

$= \sup\left\{|x^n - g(x)| = \begin{cases} x^n, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}\right\}$

$= 1$ (since $x^n \to 1$ as $x \to 1^-$)

$\|g_n - g\|_{[0,1]} \not\to 0$, $\therefore$ $g_n \not\rightrightarrows g$ on $[0,1]$.

(c) eg 8.1.2 (c). $f_n(x) = \frac{x^2+nx}{n}$, $f(x)=x$ on $\mathbb{R}$

But $f_n(x) - f(x) = \frac{x^2}{n}$ is not bounded on $\mathbb{R}$.

$\therefore$ $\|f_n - f\|_{\mathbb{R}}$ doesn't define

But $f_n(x) - f(x) = \frac{x^2}{n}$ is bounded on $[0,8]$, and

$\|f_n - f\|_{[0,8]} = \sup\{|\frac{x^2}{n}|, x \in [0,8]\} = \frac{64}{n}$

$\to 0$ as $n \to \infty$

$\therefore$ $f_n \rightrightarrows f$ on $[0,8]$ (but not on $\mathbb{R}$)

(d) eg 8.1.2(d) $F_n(x) = \frac{1}{n}\sin(n(x+1))$, $F(x)=0$ on $\mathbb{R}$.

$|F_n(x) - F(x)| \le \frac{1}{n}$, $\forall x \in \mathbb{R}$

$\Rightarrow$ $\|F_n - F\|_{\mathbb{R}} \le \frac{1}{n}$ (in fact $\|F_n - F\| = \frac{1}{n}$ (Ex!))

$\to 0$ as $n \to \infty$

$\therefore$ $F_n \rightrightarrows F$ on $\mathbb{R}$.

(e) $A = [0,1]$, $G_n(x) = x^n(1-x)$.

Clearly $G_n(x) \to 0$ $\forall x \in [0,1]$ (Ex!)

$\therefore$ G_n converges pointwisely to $G(x) \equiv 0$ on $A = [0,1]$.

To see whether G_n converges uniformly to G on $[0,1]$,

we calculate $\|G_n - G\|_{[0,1]}$:

$\forall x \in [0,1], \quad |G_n(x) - G(x)| = x^n(1-x) \geq 0$

which is 0 at $x = 0, 1$.

$\therefore$ For interior max: $x \neq 0, 1$

$$0 = (x^n(1-x))' = nx^{n-1}(1-x) - x^n$$

$$= x^{n-1}(n - (n+1)x)$$

$G(x) \geq 0$ on $[0,1]$ & $G(0) = G(1) = 0$

$$\Rightarrow \quad x = \frac{n}{n+1} \text{ (only critical pt, hence "maximum")}$$

and
$$\|G_n - G\|_{[0,1]} = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{n+1}$$

Note that $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$, we have

$$\|G_n - G\|_{[0,1]} \to 0 \text{ as } n \to \infty$$

$\therefore$ G_n converges uniformly to G on $[0,1]$. #

__Thm 8.1.10__ (__Cauchy Criterion for Uniform Convergence__)

Let f_n be a seq. of bounded functions on A. Then f_n __converges uniformly__ to a bounded function f on A

$\Leftrightarrow$ $\forall \varepsilon > 0, \exists H(\varepsilon) \in \mathbb{N}$ s.t. $\forall m, n \geq H(\varepsilon)$,

$$\|f_m - f_n\|_A < \varepsilon.$$

Pf: ($\Rightarrow$) f_n converges uniformly to f on A (both f_n, f bdd)

$\Rightarrow \quad \|f_n - f\|_A \to 0$ (Lemma 8.1.8)

$\therefore \ \forall \varepsilon > 0, \ \exists K(\varepsilon/2) \in \mathbb{N}$ s.t.

if $n \geq K(\varepsilon/2)$, then $\|f_n - f\|_A < \varepsilon/2$.

Hence letting $H(\varepsilon) = K(\varepsilon/2)$, we have

$\forall m, n \geq H(\varepsilon), \quad \|f_n - f\|_A < \varepsilon/2$ & $\|f_m - f\|_A < \varepsilon/2$

$$\Rightarrow \|f_m - f_n\|_A = \sup\{|f_m(x) - f_n(x)| : x \in A\}$$

$$\leq \sup\{|f_m(x) - f(x)| + |f_n(x) - f(x)| : x \in A\}$$

$$\leq \sup\{|f_m(x) - f(x)| : x \in A\} + \sup\{|f_n(x) - f(x)| : x \in A\}$$

$$= \|f_m - f\|_A + \|f_n - f\|_A < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

($\Leftarrow$) Conversely, if $\forall \varepsilon > 0, \exists H(\varepsilon) > 0$ s.t.

$\forall m, n \geq H(\varepsilon), \ \|f_m - f_n\|_A < \varepsilon$.

Then $\forall x \in A, \quad |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_A < \varepsilon$ —— ($*$)

$\Rightarrow (f_n(x))$ is a Cauchy sequence.

By completeness of $\mathbb{R}$ (Thm 3.5.5), $f_n(x)$ is convergent.

Since the limit depends on x, we denote it by

$f(x) \overset{\text{def}}{=} \lim_{n \to \infty} f_n(x)$. ($f(x)$ is the pointwise limit of $f_n(x)$)

Then letting $m \to \infty$ in $(*)$, we have

$$|f(x) - f_n(x)| \leq \varepsilon, \ \forall \ x \in A.$$

i.e. $\forall \varepsilon > 0$, $\exists \ H(\varepsilon) \in \mathbb{N}$ s.t.

$$\text{if } n \geq H(\varepsilon), \ |f(x) - f_n(x)| \leq \varepsilon, \ \forall \ x \in A.$$

Since $\varepsilon > 0$ is arbitrary, this shows that f_n converges uniformly to f on A. ※

§8.2 Interchange of Limits

Eg 8.2.1

(a) (Eg 8.1.2(b)) $g_n(x) = x^n$ on $[0,1]$

$$g_n(x) \to g(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \quad \text{pointwise}$$

$\uparrow$ discontinuous

$$\lim_{n\to\infty} \lim_{\substack{x\to 1 \\ (x\neq 1)}} g_n(x) = \lim_{n\to\infty} \lim_{\substack{x\to 1 \\ (x\neq 1)}} x^n = \lim_{n\to\infty} 1 = 1$$

$$\lim_{\substack{x\to 1 \\ (x\neq 1)}} \lim_{n\to\infty} g_n(x) = \lim_{\substack{x\to 1 \\ (x\neq 1)}} g(x) = 0 \qquad (\text{since } g(x) = 0, \forall x < 1)$$

$$\therefore \lim_{n\to\infty} \lim_{\substack{x\to 1 \\ (x\neq 1)}} g_n(x) \neq \lim_{\substack{x\to 1 \\ (x\neq 1)}} \lim_{n\to\infty} g_n(x)$$

$\therefore$ Can't change limits for "pointwise convergence".

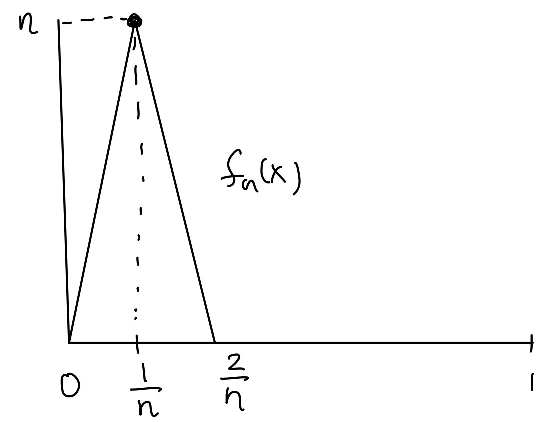
(b) (Same example) $g_n'(x) = n x^{n-1}$

$$g'(x) = \begin{cases} 0 & , 0 \le x < 1 \\ \text{doesn't exist} & , x = 1 \end{cases}$$

$\therefore$ "Pointwise limit" of sequence of differentiable functions may not differentiable.

(c)
$$f_n(x) = \begin{cases} n^2 x & , \quad 0 \le x \le \frac{1}{n} \\ -n^2\left(x - \frac{2}{n}\right) & , \quad \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & , \quad \frac{2}{n} \le x \le 1 \end{cases}$$
$(n \ge 2)$

(well-defined at $x = \frac{1}{n}$ and $x = \frac{2}{n}$)



It is easy to prove

$\lim_{n\to\infty} f_n(x) = 0$, $\forall x \in [0,1]$

$\therefore$ $f_n \to 0$ pointwisely

As f_n is cts, f_n is Riemann integrable and
$$\int_0^1 f_n = 1, \quad \forall n \ge 2.$$
$$\therefore \quad 1 = \lim_{n\to\infty} \int_0^1 f_n \ne \int_0^1 \lim_{n\to\infty} f_n = 0.$$

$\therefore$ Integral of pointwise limit $\ne$ limit of integrals.

(d) Let $h_n(x) = 2nx e^{-nx^2}$, $x \in [0,1]$.

Then $\int_0^1 h_n = \int_0^1 2nxe^{-nx^2} dx = \int_0^1 (-e^{-nx^2})' \, dx$
$$= -e^{-nx^2}\Big|_0^1 = 1 - e^{-n}$$

$\therefore$ $\lim_{n\to\infty} \int_0^1 h_n = 1$

But $\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} 2nxe^{-nx^2} = 0 \quad \forall x \in [0,1]$ (Ex!)

$\therefore$ $\int_0^1 \lim_{n\to\infty} h_n = 0 \ne \lim_{n\to\infty} \int_0^1 h_n$.

Interchange of Limit and Continuity

Thm 8.2.2 Let
- $f_n : A \to \mathbb{R}$ seq of continuous functions
- $f : A \to \mathbb{R}$
- $f_n \rightrightarrows f$ on A (converges uniformly)

Then f is continuous on A.

(i.e. uniform limit of continuous functions is continuous)

Pf: $f_n \rightrightarrows f$ on A

$\Leftrightarrow \quad \|f_n - f\|_A \to 0$

$\Rightarrow \quad \forall \varepsilon > 0, \exists H = H(\frac{\varepsilon}{3}) > 0$ s.t.

if $n \geq H$, $\|f_n - f\|_A < \frac{\varepsilon}{3}$

$\|f_n - f\|_A = \sup\{|f_n(x) - f(x)| : x \in A\}$

Now if $c \in A$, then $\forall x \in A$

$$|f(x) - f(c)| \leq |f(x) - f_H(x)| + |f_H(x) - f_H(c)| + |f_H(c) - f(c)|$$

$$\leq \|f_H - f\|_A + |f_H(x) - f_H(c)| + \|f_H - f\|_A$$

$$< \frac{2\varepsilon}{3} + |f_H(x) - f_H(c)|$$

Since f_H is continuous, $\exists \delta_\varepsilon(c) > 0$ such that

if $|x - c| < \delta_\varepsilon$, then $|f_H(x) - f_H(c)| < \varepsilon/3$.

Therefore, we have proved that

$\forall \varepsilon > 0$, $\exists \delta_\varepsilon(c) > 0$ s.t.

if $|x - c| < \delta_\varepsilon$,

$$|f(x) - f(c)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\therefore$ f is continuous at c.

Since $c \in A$ is arbitrary, f is continuous on A. #

Interchange of Limit and Derivative

Thm 8.2.3 Let

- I be a bounded interval ($a<b$ finite numbers, $[a,b], (a,b], [a,b), (a,b)$)
- $f_n : I \to \mathbb{R}$ seq. of functions
- $\exists\, x_0 \in I$ such that $f_n(x_0)$ converges as $n \to +\infty$.
- f_n' exists on I (f_n' may not be continuous)
- $f_n' \rightrightarrows g$ on I for some function g (uniform convergent)

Then $\exists$ differentiable $f : I \to \mathbb{R}$

such that

- $f_n \rightrightarrows f$ on I, and
- $f' = g$ $\left(\left(\lim_{n\to\infty} f_n \right)' = \lim_{n\to\infty} f_n' \right)$

<u>Remark</u>: Since f_n' is not assumed to be continuous, f_n' may not integrable and hence the Fundamental Thm of Calculus may not applicable.

<u>Pf</u>: Let $m,n \in \mathbb{N}$, f_m' & f_n' exist

$\Rightarrow f_m - f_n$ is differentiable

Mean Value Thm $\Rightarrow$ if $x \in I$, then

$$(f_m-f_n)(x) - (f_m-f_n)(x_0) = (f_m'-f_n')(y)\,(x-x_0)$$

for some y between x & x_0,

where x_0 is the pt such that $(f_n(x_0))$ converges.

$$\therefore\ f_m(x) - f_n(x) = f_m(x_0) - f_n(x_0) + \left(f_m'(y) - f_n'(y)\right)(x-x_0)$$

$$\Rightarrow\ |f_m(x) - f_n(x)| \le |f_m(x_0) - f_n(x_0)| + |f_m'(y) - f_n'(y)|\,|x-x_0|$$

$$\le |f_m(x_0) - f_n(x_0)| + \|f_m' - f_n'\|_I\,(b-a),$$

where $a<b$ are the endpts of I.

Taking sup over $x \in I$, we have

$$\|f_m - f_n\|_I \le |f_m(x_0) - f_n(x_0)| + \|f_m' - f_n'\|_I\,(b-a) \quad \text{——} (*),$$

Since $f_n' \rightrightarrows g$, Cauchy criterion for uniform convergence (Thm 8.1.10) implies

$\forall \varepsilon > 0$, $\exists\ H_1 = H(\frac{\varepsilon}{2(b-a)}) \in \mathbb{N}$ such that

$$\| f_m' - f_n' \|_I < \frac{\varepsilon}{2(b-a)} \quad , \forall\ m, n \geq H_1$$

Since $(f_n(x_0))$ converges, Cauchy criterion for convergence of sequence (Thm 3.5.5) implies

$\forall \varepsilon > 0$, $\exists\ H_2 = H(\frac{\varepsilon}{2}) \in \mathbb{N}$ such that

$$| f_m(x_0) - f_n(x_0) | < \frac{\varepsilon}{2} \quad , \forall\ m, n \geq H_2$$

Hence using $(*)_1$,

$\forall \varepsilon > 0$, $\exists\ H = \max\{H_1, H_2\} \in \mathbb{N}$ such that if $m, n \geq H$,

$$\| f_m - f_n \|_I < \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)} (b-a) = \varepsilon$$

Then Cauchy Criterion for uniform convergence again implies

$$f_m \rightrightarrows f \quad \text{for some function } f : I \to \mathbb{R}$$

(converges uniformly to some f)

Next, we need to show that f is differentiable and

$$f' = g.$$

(To be cont'd next time)