§7.4 The Darboux Integral

Def (Upper and Lower Sums)

Let
- $f : [a,b] \to \mathbb{R}$ bounded
- $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ partition of $[a,b]$
- $m_k = \inf\{ f(x) : x \in [x_{k-1}, x_k] \}$ (exist because of "bddness")

  $M_k = \sup\{ f(x) : x \in [x_{k-1}, x_k] \}$

The
- lower sum of f corresponding to $\mathcal{P}$ is defined to be

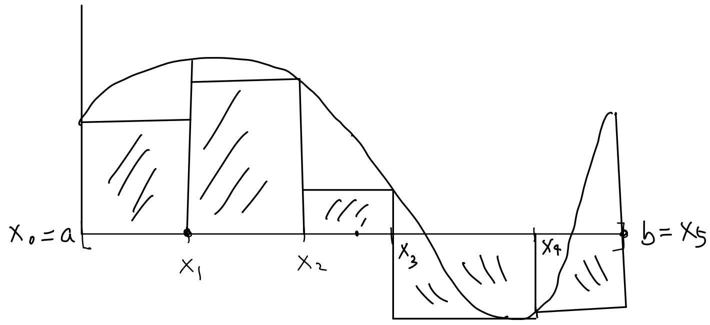
$$L(f; \mathcal{P}) = \sum_{k=1}^{n} m_k (x_k - x_{k-1}) \quad ;$$

- upper sum of f corresponding to $\mathcal{P}$ is defined to be

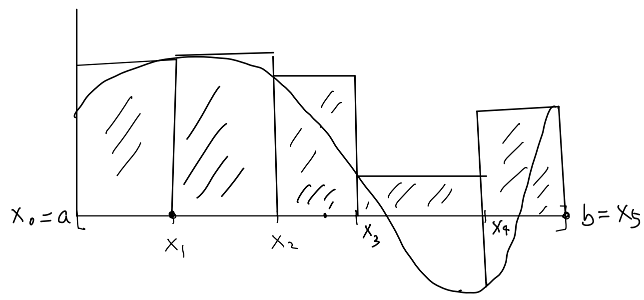
$$U(f; \mathcal{P}) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

Remarks (i) upper and lower sums are not Riemann sums in general, (because m_k, M_k may not attained at any point in $[x_{k-1}, x_k]$) unless the function f is cts.

(ii) On one hand, $L(f; \mathcal{P})$ and $U(f; \mathcal{P})$ are simpler because they do not involve the infinite many possibility of tags. But on the other hand, inf and sup are harder to handle than values of a function.



lower sum $L(f;\mathcal{P})$



upper sum $U(f;\mathcal{P})$

Lemma 7.4.1 If $f:[a,b]\to\mathbb{R}$ is bounded and $\mathcal{P}$ is a partition of $[a,b]$.

Then $L(f;\mathcal{P}) \le U(f;\mathcal{P})$

Pf: (Easy)
$$m_k = \inf_{[x_{k-1},x_k]} f \le \sup_{[x_{k-1},x_k]} f = M_k$$

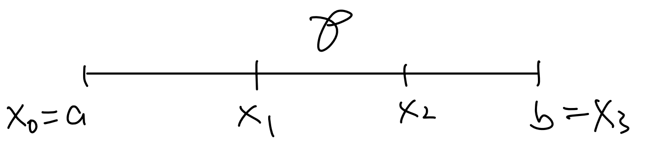
$$\Rightarrow \quad L(f;\mathcal{P}) = \sum_k m_k(x_k - x_{k-1}) \le \sum_k M_k(x_k - x_{k-1}) = U(f;\mathcal{P})$$

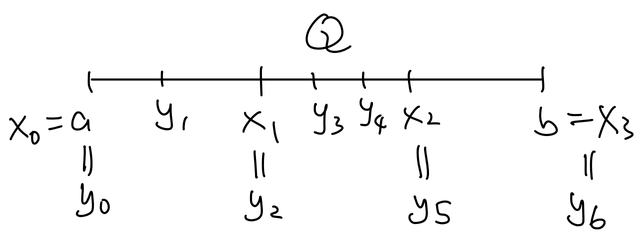
Def: If $\mathcal{P}$, Q are partitions of $[a,b]$ and $\mathcal{P}\subset Q$, then we say that Q is a refinement of $\mathcal{P}$.

Remark: If $\mathcal{P}=(x_0,x_1,\cdots,x_n)$ and $Q=(y_0,y_1,\cdots,y_m)$,

then Q is a refinement of $\mathcal{P}$ iff $x_k\in\mathcal{P}$, $\forall k=0,1,\cdots,n \Rightarrow x_k\in Q$

(i.e. $x_k = y_l$ for some $l=0,1,\cdots,m$)





In other words, subinterval $[x_{k-1}, x_k]$ of $\mathcal{P}$ is further subdivided in Q: $\quad [x_{k-1}, x_k] = [y_{j-1}, y_j] \cup \cdots \cup [y_{h-1}, y_h]$.

Lemma 7.4.2 If
- $f : [a,b] \to \mathbb{R}$ is bounded
- $\mathcal{P}$ is a partition of $[a,b]$
- Q is a refinement of $\mathcal{P}$.

Then $L(f;\mathcal{P}) \le L(f;Q)$ and $U(L;Q) \le U(f,\mathcal{P})$

Pf: Special case Q is a refinement of $\mathcal{P}$ by adjoining one point.

Let $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ and

$$Q = (x_0, x_1, \cdots, x_{k-1}, z, x_k, \cdots x_n)$$

Then
$$m_k' = \inf\{f(x) : x \in [x_{k-1}, z]\}$$
$$\ge \inf\{f(x) : x \in [x_{k-1}, x_k]\} = m_k$$

&
$$m_k'' = \inf\{f(x) : x \in [z, x_k]\}$$
$$\ge \inf\{f(x) : x \in [x_{k-1}, x_k]\} = m_k$$

$$\Rightarrow \quad L(f;\mathcal{P}) = \sum_{i \ne k} m_i(x_i - x_{i-1}) + m_k(x_k - x_{k-1})$$
$$= \sum_{i \ne k} m_i(x_i - x_{i-1}) + m_k(z - x_{k-1}) + m_k(x_k - z)$$
$$\le \sum_{i \ne k} m_i(x_i - x_{i-1}) + m_k''(z - x_{k-1}) + m_k'(x_k - z)$$
$$= L(f;Q)$$

Similarly $U(f;\mathcal{P}) \geq U(f;Q)$ (ex!)

<u>General Case</u>

If Q is a refinement of $\mathcal{P}$, then Q can be obtain from by adjoining a finite number of points to $\mathcal{P}$ <u>one at a time</u>.

Hence, repeating the special case (or using induction), we have $L(f;\mathcal{P}) \leq L(f;Q)$

and $U(f;Q) \leq U(f;\mathcal{P})$ ⋕

<u>Lemma 7.4.3</u> Let • $f:[a,b] \to \mathbb{R}$ be <u>bounded</u>.

Then $L(f;\mathcal{P}_1) \leq U(f;\mathcal{P}_2)$

for any partitions $\mathcal{P}_1, \mathcal{P}_2$ of $[a,b]$.

Pf: Let $Q = \mathcal{P}_1 \cup \mathcal{P}_2$.

Then Q is a refinement of $\mathcal{P}_1$ and also of $\mathcal{P}_2$.

Hence Lemma 7.4.1 & Lemma 7.4.2

$\Rightarrow$ $L(f;\mathcal{P}_1) \leq L(f;Q) \leq U(f;Q) \leq U(f;\mathcal{P}_2)$

⋕

<u>Notation</u>: Let $\mathcal{P}([a,b])$ = set of partitions of $[a,b]$.

<u>Def 7.4.4</u> Let $f:[a,b]\to\mathbb{R}$ be <u>bounded</u>.

The <u>lower integral of f on I</u> is the number

$$L(f) = \sup\{L(f;\mathcal{P}) : \mathcal{P}\in\mathcal{P}[a,b]\}$$

and the <u>upper integral of f on I</u> is the number

$$U(f) = \inf\{U(f;\mathcal{P}) : \mathcal{P}\in\mathcal{P}[a,b]\}$$

<u>Thm 7.4.5</u> Let $f:[a,b]\to\mathbb{R}$ be bounded. Then $L(f)$ and $U(f)$ of f on $[a,b]$ exist and $L(f)\le U(f)$

<u>Pf</u>: • <u>$L(f)$ and $U(f)$ exist</u>

f bounded $\Rightarrow$ $m_I = \inf\{f(x) : x\in I=[a,b]\}$ &

$M_I = \sup\{f(x) : x\in I=[a,b]\}$ exist

It is clear that $\forall\, \mathcal{P}\in\mathcal{P}([a,b])$

$$m_I(b-a)\le L(f;\mathcal{P})\le U(f;\mathcal{P})\le M_I(b-a)$$

$\therefore$ $L(f)$ and $U(f)$ exist

(and satisfy $m_I(b-a)\le L(f)$ & $U(f)\le M_I(b-a)$)

• $L(f) \leq U(f)$

By Lemma 7.4.3,

$$L(f;\mathcal{P}_1) \leq U(f;\mathcal{P}_2) \quad \text{for any partitions } \mathcal{P}_1 \ \& \ \mathcal{P}_2$$

Fixing $\mathcal{P}_2$ and letting $\mathcal{P}_1$ runs through $\mathscr{P}([a,b])$, we have

$$L(f) = \sup\{L(f;\mathcal{P}_1) : \mathcal{P}_1 \in \mathscr{P}([a,b])\} \leq U(f;\mathcal{P}_2).$$

Then letting $\mathcal{P}_2$ runs through $\mathscr{P}([a,b])$, we have

$$L(f) \leq \inf\{U(f;\mathcal{P}_2) : \mathcal{P}_2 \in \mathscr{P}([a,b])\} = U(f)$$ ##

<u>Def 7.4.6</u> Let $f : [a,b] \to \mathbb{R}$ be <u>bounded</u>. Then f is said to be <u>Darboux integrable</u> on $[a,b]$ if $\underline{L(f) = U(f)}$.

In this case, the <u>Darboux integral</u> of f over $[a,b]$ is defined to be the value $L(f) = U(f)$.

<u>Remark</u>: We'll use the same notation $\int_a^b f$ or $\int_a^b f(x)dx$ for Darboux integral (since it is equal to the Riemann integral (Thm 7.4.11))

Eg 7.4.7

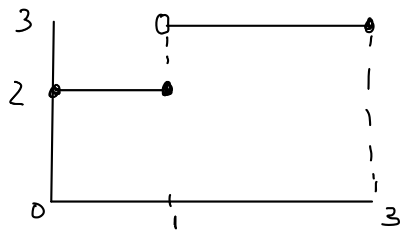
(a) A constant function is Darboux integrable

In fact, if $f(x) \equiv c$ on $[a,b]$ & $\mathcal{P}$ is any partition of $[a,b]$,

then $$L(f;\mathcal{P}) = c(b-a) = U(f;\mathcal{P}) \quad (\text{Ex } 7.4.2)$$

$$\therefore \quad L(f) = c(b-a) = U(f) \quad ※$$

(b) $g:[0,3] \to \mathbb{R}$ defined by $g(x) = \begin{cases} 3, & 1 < x \leq 3 \\ 2, & 0 \leq x \leq 1 \end{cases}$ (eg 7.1.4(b))



(is (Riemann) integrable & $\int_0^3 g = 8$.)

Using Darboux's approach, we only need to prove

$$L(f) = U(f)$$

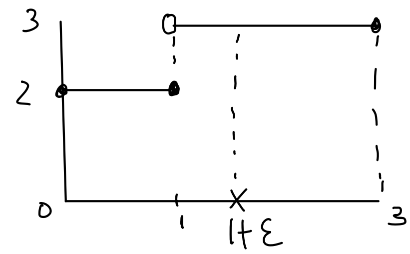
No need to check whether they exist.

As $L(f) = \sup\{\text{of something}\}$ &

$U(f) = \inf\{\text{of something}\}$

we only need to find sequence/family of partitions that can prove the required result, no need to consider all partitions.

g is clearly bounded.



$\forall \varepsilon > 0$, consider the partition

$\mathcal{P}_\varepsilon = (0, 1, 1+\varepsilon, 3)$

Then

$$U(g; \mathcal{P}_\varepsilon) = 2 \cdot (1-0) + 3 \cdot (1+\varepsilon - 1) + 3 \cdot (3 - (1+\varepsilon))$$

$$= 2 + 3\varepsilon + 6 - 3\varepsilon = 8$$

$\Rightarrow \quad U(g) \leq 8 \qquad \left(U(g) = \inf \{ U(g; \mathcal{P}) : \mathcal{P} \in \mathcal{P}([0,3]) \} \right)$

And

$$L(g; \mathcal{P}_\varepsilon) = 2 \cdot (1-0) + 2 \cdot (1+\varepsilon - 1) + 3 \cdot (3 - (1+\varepsilon))$$

$$\left(\inf \{ g(x) : x \in [1, 1+\varepsilon] \} = 2 \right)$$

$$= 2 + 2\varepsilon + 6 - 3\varepsilon = 8 - \varepsilon$$

$\Rightarrow \quad 8 - \varepsilon \leq L(g) \qquad \left(L(g) = \sup \{ L(g; \mathcal{P}) : \mathcal{P} \in \mathcal{P}([0,3]) \} \right)$

Hence, Thm 7.4.5 $\Rightarrow$

$$8 - \varepsilon \leq L(g) \leq U(g) \leq 8$$

Since $\varepsilon > 0$ is arbitrary, we have

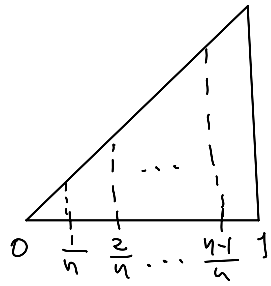
$$L(g) = U(g) = 8$$

$\therefore$ g is Darboux integrable & $\int_a^b g = 8$

(Easier than "Riemann")

(c) $h(x) = x$ on $[0,1]$ is integrable

h is clearly bounded.



Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$.

Then $U(h; P_n) = \frac{1}{n} \cdot (\frac{1}{n} - 0) + \frac{2}{n} \cdot (\frac{2}{n} - \frac{1}{n}) + \cdots + 1 \cdot (1 - \frac{n-1}{n})$

$$= \frac{1}{n^2}(1 + 2 + \cdots + n) = \frac{n(n+1)}{2n^2} = \frac{1}{2}(1 + \frac{1}{n})$$

and $L(h; P_n) = 0 \cdot (\frac{1}{n} - 0) + \frac{1}{n} \cdot (\frac{2}{n} - \frac{1}{n}) + \cdots + \frac{n-1}{n} \cdot (1 - \frac{n-1}{n})$

$$= \frac{1}{n^2}(1 + 2 + \cdots + (n-1)) = \frac{n(n-1)}{2n^2} = \frac{1}{2}(1 - \frac{1}{n})$$

$\therefore$ $\frac{1}{2}(1 - \frac{1}{n}) \leq L(h) \leq U(h) \leq \frac{1}{2}(1 + \frac{1}{n})$

Letting $n \to \infty$, we have $L(h) = U(h) = \frac{1}{2}$

$\therefore$ $h(x) = x$ is Darboux integrable on $[0,1]$

& $\int_a^b h = \frac{1}{2}$.

(d) (Eg 7.2.2 (b), not integrable)

Dirichlet function $f(x) = \begin{cases} 1, & \text{if } x \text{ rational}, x \in [0,1] \\ 0, & \text{if } x \text{ irrational}, x \in [0,1]. \end{cases}$

[To prove non-integrable, we need to consider all partitions, as a sequence/family of partitions can only provide upper bound for $U(f)$ & lower bound for $L(f)$; not good enough to see $U(f) > L(f)$.]

f is clearly bounded: $0 \le f \le 1$.

Let $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ be a partition of $[0,1]$.

Then for each subinterval $[x_{k-1}, x_k]$,

$\exists$ rational $r_k \in [x_{k-1}, x_k]$ and

irrational $t_k \in [x_{k-1}, x_k]$

$$\Rightarrow \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = f(r_k) = 1 \quad \&$$

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = f(t_k) = 0$$

$$\therefore \quad U(f;\mathcal{P}) = \sum_k M_k(x_k - x_{k-1}) = \sum_k (x_k - x_{k-1}) = 1, \quad \forall \mathcal{P}$$

$$\Rightarrow \quad U(f) = \inf\{U(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}([0,1])\} = 1$$

And $L(f;\mathcal{P}) = \sum_k m_k(x_k - x_{k-1}) = 0, \quad \forall \mathcal{P}$

$$\Rightarrow \quad L(f) = \sup\{L(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}([0,1])\} = 0.$$

$$\therefore \quad U(f) = 1 > 0 = L(f)$$

f is not Darboux integrable.

Thm 7.4.8 (Integrability Criterion)

Let $f: [a,b] \to \mathbb{R}$ be bounded.

Then f is Darboux integrable

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists$ partition $\mathcal{P}_\varepsilon$ of $[a,b]$ such that

$$U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon.$$

Pf: ($\Rightarrow$) f Darboux integrable

$$\Rightarrow L(f) = U(f).$$

Now $\forall \varepsilon > 0$, $\exists$ partition $\mathcal{P}_1$ of $[a,b]$ s.t.

$$L(f) - \frac{\varepsilon}{2} < L(f;\mathcal{P}_1) \qquad \left(\text{as } L(f) = \sup\{L(f;\mathcal{P}) : \mathcal{P} \in \mathcal{P}([0,1])\}\right),$$

and partition $\mathcal{P}_2$ of $[a,b]$ s.t.

$$U(f;\mathcal{P}_2) < U(f) + \frac{\varepsilon}{2} \qquad \left(\text{as } U(f) = \inf\{U(f;\mathcal{P}) : \mathcal{P} \in \mathcal{P}([0,1])\}\right)$$

Then the partition $\mathcal{P}_\varepsilon = \mathcal{P}_1 \cup \mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ & $\mathcal{P}_2$, and hence by lemmas 7.4.1 & 7.4.2

$$L(f) - \frac{\varepsilon}{2} < L(f;\mathcal{P}_1) \le L(f;\mathcal{P}_\varepsilon)$$

$$\le U(f;\mathcal{P}_\varepsilon) \le U(f;\mathcal{P}_2) < U(f) + \frac{\varepsilon}{2}$$

$$\Rightarrow U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < U(f) + \frac{\varepsilon}{2} - \left(L(f) - \frac{\varepsilon}{2}\right)$$

$$= \varepsilon \qquad (\text{as } U(f) = L(f))$$

($\Leftarrow$) For the converse, we observe $\forall$ partition $\mathcal{P}_\varepsilon$,

$$L(f;\mathcal{P}_\varepsilon) \le L(f) \quad \& \quad U(f) \le U(f;\mathcal{P}_\varepsilon)$$

$$\therefore\ 0 \le U(f) - L(f) \le U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, $U(f) = L(f)$

$\therefore$ f is Darboux integrable. $\#$

> <u>Cor 7.4.9</u> Let $f : [a,b] \to \mathbb{R}$ bounded
>
> If $\mathcal{P}_n$, $n = 1, 2, \cdots$, is a sequence of partitions of I s.t.
>
> $$\lim_{n\to\infty} \left(U(f;\mathcal{P}_n) - L(f;\mathcal{P}_n)\right) = 0,$$
>
> then f is (Darboux) <u>integrable</u> &
>
> $$\int_a^b f = \lim_{n\to\infty} L(f;\mathcal{P}_n) = \lim_{n\to\infty} U(f;\mathcal{P}_n)$$

Pf: $\forall \varepsilon > 0$, $\exists\, n_\varepsilon > 0$ s.t.

$$0 \le U(f;\mathcal{P}_n) - L(f;\mathcal{P}_n) < \varepsilon, \quad \forall\, n \ge n_\varepsilon$$

Just pick one of the $\mathcal{P}_n$, $n \ge n_\varepsilon$ (says $\mathcal{P}_{n_\varepsilon}$) as $\mathcal{P}_\varepsilon$

and use the Integrability Criterion (Thm 7.4.8) $\#$

Thm 7.4.10 Let $f:[a,b]\to\mathbb{R}$ be either continuous or monotone.

Then f is Darboux integrable on $[a,b]$.

Pf: Let $\mathcal{P}_n = (x_0, x_1, \cdots, x_n)$ be uniform partition of $[a,b]$ s.t.

$$x_k - x_{k-1} = \frac{b-a}{n}.$$

(1) If f is continuous, then

$$M_k = \sup\{f(x) : [x_{k-1}, x_k]\} = f(v_k) \quad \text{for some } v_k \in [x_{k-1}, x_k]$$

$$m_k = \inf\{f(x) : [x_{k-1}, x_k]\} = f(u_k) \quad \text{for some } u_k \in [x_{k-1}, x_k]$$

Then

$$L(f;\mathcal{P}_n) = \sum_k m_k (x_k - x_{k-1}) = \sum_k f(u_k)(x_k - x_{k-1})$$
$$= \int_a^b \alpha_\varepsilon$$

where α_ε is the step function (& n s.t. $\frac{b-a}{n} < \delta_\varepsilon$) as in the proof of Thm 7.2.7.

and
$$U(f;\mathcal{P}_n) = \sum_k M_k (x_k - x_{k-1}) = \sum_k f(v_k)(x_k - x_{k-1})$$
$$= \int_a^b \omega_\varepsilon$$

where ω_ε is the step function (& n s.t. $\frac{b-a}{n} < \delta_\varepsilon$) as in the proof of Thm 7.2.7.

$$\Rightarrow \quad U(f;\mathcal{P}_n) - L(f;\mathcal{P}_n) = \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$$

$\therefore$ Cor 7.4.9 $\Rightarrow$ f is Darboux integrable.

(2) If f is monotone (may assume increasing).

Then $M_k = \sup\{f(x) : [x_{k-1}, x_k]\} = f(x_k)$

$m_k = \inf\{f(x) : [x_{k-1}, x_k]\} = f(x_{k-1})$

and

$$L(f;\mathcal{P}_n) = \sum_k f(x_{k-1})(x_k - x_{k-1}) = \int_a^b \alpha$$

$$U(f;\mathcal{P}_n) = \sum_k f(x_k)(x_k - x_{k-1}) = \int_a^b \omega$$

where α, ω are functions as in the proof of Thm 7.2.8

$$\Rightarrow \quad U(f;\mathcal{P}_n) - L(f;\mathcal{P}_n) = \int_a^b (\omega - \alpha)$$

$$= \frac{b-a}{n}(f(b) - f(a))$$

$$\longrightarrow 0 \quad \text{as } n \to \infty$$

$\therefore$ Cor 7.4.9 $\Rightarrow$ f is Darboux integrable. ※

Thm 7.4.11 (Equivalence Theorem)

Let $f : [a,b] \to \mathbb{R}$. Then

f is Darboux integrable $\Leftrightarrow$ f is Riemann integrable

In this case, the integrals equal.

Pf: ($\Rightarrow$) Assume f is Darboux integrable

By Thm 7.4.8 (Integrability Criterion),

$\forall \varepsilon > 0$, $\exists$ partition $\mathcal{P}_\varepsilon$ of $[a,b]$ s.t.

$$U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon.$$

If $\mathcal{P}_\varepsilon = \{[x_{k-1}, x_k]\}_{k=1}^{n}$, define step functions α_ε & ω_ε

s.t. $\alpha_\varepsilon(x) = m_k = \inf_{[x_{k-1},x_k]} f$, $\forall x \in [x_{k-1}, x_k)$ if $k = 1,\cdots,n-1$, $[x_{n-1}, x_n]$ if $k = n$

and $\omega_\varepsilon(x) = M_k = \sup_{[x_{k-1},x_k]} f$, $\forall x \in [x_{k-1}, x_k)$ if $k = 1,\cdots,n-1$, $[x_{n-1}, x_n]$ if $k = n$

Then $\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x)$ $\forall x \in [a,b]$.

and
$$\int_a^b \alpha_\varepsilon = \sum_k m_k (x_k - x_{k-1}) = L(f;\mathcal{P}_\varepsilon)$$

$$\int_a^b \omega_\varepsilon = \sum_k M_k (x_k - x_{k-1}) = U(f;\mathcal{P}_\varepsilon)$$

$$\Rightarrow \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon$$

$\therefore$ Squeeze Thm 7.2.1 $\Rightarrow$ $f \in \mathcal{R}[a,b]$.

($\Leftarrow$) If $f \in \mathcal{R}[a,b]$ with $A = \int_a^b f$

Then f is bounded on $[a,b]$ and

$\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ s.t. if $\dot{\mathcal{P}}$ satisfies $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$,

then $\quad |S(f;\dot{\mathcal{P}}) - A| < \varepsilon$.

Let $\mathcal{P} = (x_0, x_1, \cdots, x_n)$ be a partition with $\|\mathcal{P}\| < \delta_\varepsilon$.

By definition of $M_k = \sup_{[x_{k-1}, x_k]} f$, $\exists t_k \in [x_{k-1}, x_k]$

such that $\quad f(t_k) > M_k - \frac{\varepsilon}{b-a}$.

Similarly, $\exists t_k' \in [x_{k-1}, x_k]$ s.t.

$$f(t_k') < m_k + \frac{\varepsilon}{b-a}, \quad \text{where } m_k = \inf_{[x_{k-1}, x_k]} f$$

Then the tagged partition $\dot{\mathcal{P}} = \{[x_{k-1}, x_k], t_k\}_{k=1}^n$ has

Riemann sum
$$\begin{aligned} S(f;\dot{\mathcal{P}}) &= \sum_{k=1}^n f(t_k)(x_k - x_{k-1}) \\ &> \sum_{k=1}^n \left(M_k - \frac{\varepsilon}{b-a}\right)(x_k - x_{k-1}) \\ &= \sum_{k=1}^n M_k(x_k - x_{k-1}) - \frac{\varepsilon}{b-a}\sum_{k=1}^n (x_k - x_{k-1}) \\ &= U(f;\mathcal{P}) - \varepsilon \end{aligned}$$

Using $|S(f;\dot{\mathcal{P}}) - A| < \varepsilon$, we have

$$U(f;\mathcal{P}) < S(f;\dot{\mathcal{P}}) + \varepsilon < A + 2\varepsilon.$$

Hence $\quad U(f) < A + 2\varepsilon.$

Since $\varepsilon > 0$ is arbitrary, $\quad U(f) \le A.$

Similarly for the tagged partition $\dot{\mathcal{P}}' = \{[x_{k-1}, x_k], t_k'\}_{k=1}^{n}$,

$$\begin{aligned} S(f;\dot{\mathcal{P}}') &= \sum_{k=1}^{n} f(t_k')(x_k - x_{k-1}) \\ &< \sum_{k=1}^{n} \left(m_k + \frac{\varepsilon}{b-a}\right)(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) + \frac{\varepsilon}{b-a}\sum_{k=1}^{n}(x_k - x_{k-1}) \\ &= L(f;\mathcal{P}) + \varepsilon \end{aligned}$$

$\Rightarrow \quad L(f;\mathcal{P}) > S(f;\dot{\mathcal{P}}') - \varepsilon > A - 2\varepsilon.$

$\Rightarrow \quad L(f) > A - 2\varepsilon, \quad \forall \varepsilon > 0$

$\Rightarrow \quad L(f) \ge A.$

Therefore $\quad A \le L(f) \le U(f) \le A$

$\Rightarrow \quad f$ is Darboux integrable,

and the Darboux integral $= A$. #

§7.5 Approximate Integration (Omitted)