Thm 7.3.14 (**Composition Theorem**)

Let $\begin{cases} \bullet\ f \in \mathcal{R}[a,b] \text{ with } f([a,b]) \subset [c,d], \\ \bullet\ \varphi : [c,d] \to \mathbb{R} \text{ continuous} \end{cases}$

$$\left(\begin{tikzcd} {[a,b]} \arrow[r, "f"] \arrow[rr, bend right, "\varphi\circ f"'] & {[c,d]} \arrow[r, "\varphi"] & \mathbb{R} \end{tikzcd} \right)$$

Then $\varphi \circ f \in \mathcal{R}[a,b]$.

("φ cts" is needed, see ex. 7.3.22)

Pf: Let D = set of discontinuity of f on $[a,b]$,

D_1 = set of discontinuity of $\varphi\circ f$ on $[a,b]$.

If $u \in [a,b] \setminus D$, then f is continuous at u,

Since φ is cts, $\varphi\circ f$ is also continuous at u.

$\therefore$ $u \in [a,b] \setminus D_1$

Therefore $[a,b] \setminus D \subset [a,b] \setminus D_1$,

and hence $D_1 \subset D$.

Note that $f \in \mathcal{R}[a,b]$. Lebesgue's Integrable Criterion

$\Rightarrow$ D is of measure zero.

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists$ countable collection of open intervals $\{I_k\}_{k=1}^{n}$

s.t. $D \subset \bigcup_{k=1}^{\infty} I_k$ & $\sum_{k=1}^{\infty} \text{length}(I_k) \leq \varepsilon$.

Since $D_1 \subset D$, we have

$$D_1 \subset \bigcup_{k=1}^{\infty} I_k \quad \& \quad \sum_{k=1}^{\infty} \text{length}(I_k) \le \varepsilon$$

$\therefore$ D_1 is also of measure zero.

Using Lebesgue's Integrability criterion again, we have

$$\varphi \circ f \in \mathcal{R}[a,b].$$

$\divideontimes$

(In this proof, we showed that a <u>subset</u> of a <u>null</u> set is also a <u>null</u> set.)

<u>Cor 7.3.15</u> If $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$

and $$\left| \int_a^b f \right| \le \int_a^b |f| \le M(b-a)$$

for any $M \ge 0$ s.t. $|f(x)| \le M$ on $[a,b]$

<u>Pf</u>: $f \in \mathcal{R}[a,b] \Rightarrow f$ is bounded

$\Rightarrow \quad |f(x)| \le M$ on $[a,b]$ for some $M > 0$.

Then $f([a,b]) \subset [-M, M]$ and

$|\cdot| : [-M, M] \to \mathbb{R}$ is continuous.

By Thm 7.3.14, $|f| \in \mathcal{R}[a,b]$

Since $-|f|(x) \le f(x) \le |f|(x)$, $\forall x \in [a,b]$, Thm 7.1.5(c) $\Rightarrow$ $-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$.

$\therefore$ $\left|\int_a^b f\right| \le \int_a^b |f|$.

Similarly, $|f|(x) \le M \quad \forall\, x \in [a,b]$

$\Rightarrow$ $\int_a^b |f| \le \int_a^b M = M(b-a)$ $\#$

Thm 7.3.16 (The Product Thm) If f & $g \in \mathcal{R}[a,b]$, then $fg \in \mathcal{R}[a,b]$.

Pf: $f \in \mathcal{R}[a,b] \Rightarrow \exists\, M>0$ s.t. $f([a,b]) \subset [-M,M]$.

and $\varphi(t) = t^2 : [-M,M] \to \mathbb{R}$ is cts

$\therefore$ $f^2 \in \mathcal{R}[a,b]$.

Similarly $g \in \mathcal{R}[a,b] \Rightarrow g^2 \in \mathcal{R}[a,b]$.

By Thm 7.1.5(b), $f, g \in \mathcal{R}[a,b] \Rightarrow f+g \in \mathcal{R}[a,b]$.

Hence $(f+g)^2 \in \mathcal{R}[a,b]$.

Therefore, Thm 7.1.5 again, $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right] \in \mathcal{R}[a,b]$ $\#$

Thm 7.3.17 (Integration by Parts)

Let
- F, G be differentiable on $[a,b]$
- $f = F'$, $g = G' \in \mathcal{R}[a,b]$

Then fG, $Fg \in \mathcal{R}[a,b]$ and

$$\int_a^b fG = FG\Big|_a^b - \int_a^b Fg$$

Pf: F, G diff on $[a,b] \Rightarrow F, G$ cts on $[a,b]$

$$\Rightarrow F, G \in \mathcal{R}[a,b] \quad \text{(Thm 7.2.7)}$$

Product Thm 7.3.16 then implies

$$fG \ \& \ Fg \in \mathcal{R}[a,b].$$

And product rule Thm 6.1.3 (c),

$$(FG)' = F'G + FG' = fG + Fg \in \mathcal{R}[a,b]$$

Fundamental Thm 7.3.1 $\Rightarrow$

$$\int_a^b (FG)' = FG\Big|_a^b$$

$$\therefore \int_a^b fG + \int_a^b Fg = FG\Big|_a^b$$

#

Thm 7.3.18 (Taylor's Thm with Remainder (Integral Form))

Suppose
- $f : [a,b] \to \mathbb{R}$
- $f', \cdots, f^{(n)}, f^{(n+1)}$ exist on $[a,b]$
- $f^{(n+1)} \in \mathcal{R}[a,b]$

Then $$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a) + R_n$$

where $$R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt.$$

Pf:
$$R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)\,(b-t)^n\,dt$$ (is defined by Product Thm)

$$= \int_a^b \left(f^{(n)}\right)'(t)\left(\frac{(b-t)^n}{n!}\right)dt$$ (Integration by Parts) Thm 7.3.17

$$= f^{(n)}(t)\,\frac{(b-t)^n}{n!}\Bigg|_a^b - \int_a^b f^{(n)}(t)\left[-\frac{(b-t)^{n-1}}{(n-1)!}\right]dt$$

$$= -\frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t)(b-t)^{n-1}\,dt$$

$$= -\frac{f^{(n)}(a)}{n!}(b-a)^n + R_{n-1}$$

$$= -\frac{f^{(n)}(a)}{n!}(b-a)^n - \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + R_{n-2}$$ (Same calculation)

$\vdots$

$$= -\left(\frac{f^{(n)}(a)}{n!}(b-a)^n + \cdots + \frac{f'(a)}{1!}(b-a)\right) + R_0$$

where $R_0 = \frac{1}{0!}\int_a^b f'(t)(b-t)^0\,dt = \int_a^b f' = f(b) - f(a)$

So we are done. $\#$