Cor 7.2.10 If $f \in \mathcal{R}[a,b]$ & $[c,d] \subset [a,b]$, then $f \in \mathcal{R}[c,d]$.

Pf: By Additivity Thm 7.2.8

$$f \in \mathcal{R}[a,b] \Rightarrow f \in \mathcal{R}[c,b] \Rightarrow f \in \mathcal{R}[c,d] \quad \#$$

Cor 7.2.11 If $f \in \mathcal{R}[a,b]$ & $a = c_0 < c_1 < \cdots < c_m = b$, then $f|_{[c_{i-1}, c_i]} \in \mathcal{R}[c_{i-1}, c_i]$ and

$$\int_a^b f = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} f$$

(Pf: By Induction)

Def: If $f \in \mathcal{R}[a,b]$ and $\alpha, \beta \in [a,b]$ with $\alpha < \beta$, we define $\int_\beta^\alpha f \overset{\text{def}}{=\!=} -\int_\alpha^\beta f$ and

$$\int_\alpha^\alpha f \overset{\text{def}}{=\!=} 0$$

Thm 7.2.13 If $f \in \mathcal{R}[a,b]$ and $\alpha, \beta, \gamma \in [a,b]$,

then $$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f \qquad \text{———} \quad (*)$$

in the sense that the existence of any two of these integrals exist implies the third integral exists & $(*)$ holds

Pf: Clearly Thm 7.2.9 & Cor 7.2.11 implies the statement that "the existence of any two of these integrals exist $\Rightarrow$ the third integral exists".

Now if any two of α, β, γ equal, then $(*)$ is trivially holds (check)

If α, β, γ are distinct, we consider

$$L(\alpha,\beta,\gamma) \stackrel{\text{def}}{=} \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f$$

$$= \int_\alpha^\beta f - \int_\gamma^\beta f - \int_\alpha^\gamma f$$

Clearly $L(\alpha,\beta,\gamma) = L(\beta,\gamma,\alpha) = L(\gamma,\alpha,\beta)$ (Check!)

$$= -L(\alpha,\gamma,\beta) = -L(\gamma,\beta,\alpha) = -L(\beta,\alpha,\gamma)$$

(eg: $L(\alpha,\beta,\gamma) = \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f$

$= -\int_\beta^\alpha f - \int_\gamma^\beta f - \int_\alpha^\gamma f = -L(\alpha,\gamma,\beta)$)

By Additivity Thm 7.2.9, if $\alpha < \gamma < \beta$,

then $L(\alpha,\beta,\gamma) = \int_\alpha^\beta f - \left(\int_\alpha^\gamma f + \int_\gamma^\beta f\right) = 0$.

By the above, we have $L(\alpha,\beta,\gamma) = 0$

for all other situations: $\gamma<\beta<\alpha$, $\beta<\alpha<\gamma$

$\gamma<\alpha<\beta$, $\alpha<\beta<\gamma$, & $\beta<\gamma<\alpha$.

Hence $\forall \alpha,\beta,\gamma$,

$$0 = L(\alpha,\beta,\gamma) = \int_\alpha^\beta f - \left(\int_\alpha^\gamma f + \int_\gamma^\beta f\right)$$

i.e. $$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$$

$\#$

§7.3 The Fundamental Theorem

Recall: A function $F: [a,b] \to \mathbb{R}$ is called an antiderivative

or a primitive of $f: [a,b] \to \mathbb{R}$ on $[a,b]$ if

$$F'(x) = f(x), \quad \forall x \in [a,b]$$

(One sided derivatives at $x=a$ & $x=b$)

Thm 7.3.1 (Fundamental Theorem of Calculus (1^{st} Form))

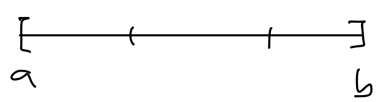
Suppose $\begin{cases} \bullet \ f, F: [a,b] \to \mathbb{R} \text{ functions}, \\ \bullet \ E = \text{finite set of } [a,b] \end{cases}$ (E for exceptional set)

such that $\begin{cases} \text{(a) } F \text{ is continuous on } [a,b], \\ \text{(b) } F'(x) = f(x) \quad \forall x \in [a,b] \setminus E, \\ \text{(c) } f \in \mathcal{R}[a,b] \end{cases}$

Then

$$\boxed{\int_a^b f = F(b) - F(a)}$$

Pf: With the finite # of points in E,



$[a,b]$ is subdivided into finite

number of subintervals such that $F'(x) = f(x)$ on the

subintervals except possibly at endpoints.

Then by Thm 7.1.3 & Thm 7.2.9, one can reduce the proof of the Thm to the case that

$E = \{a, b\}$ two end points only

i.e. $F'(x) = f(x)$, $\forall x \in (a,b)$.

(Exercise 7.3.1 of the Textbook, using F cts & $\sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = F(b) - F(a)$)

For this special case, consider any $\varepsilon > 0$.

Then $f \in \mathcal{R}[a,b]$ (assumption (c)) $\Rightarrow$

$\exists\, \delta_\varepsilon > 0$ such that

if $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ satisfies $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, (any tags t_i)

then $\left| S(f, \dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon$. ——— $(*)$

By Mean Value Thm 6.2.4, $\exists\, u_i \in (x_{i-1}, x_i)$ s.t.

$$F(x_i) - F(x_{i-1}) = F'(u_i)(x_i - x_{i-1})$$

$$= f(u_i)(x_i - x_{i-1}), \quad \forall i = 1, \cdots, n$$

since $F' = f$ exists on (a,b) (assumption (b) of the special case)

Hence $F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$

$$= \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1})$$

Define the tagged partition $\dot{\mathcal{P}}_u = \{[x_{i-1}, x_i], u_i\}_{i=1}^{n}$

(same partition with new tags).

Then $\|\dot{\mathcal{P}}_u\| < \delta_\varepsilon$ and

$$F(b) - F(a) = S(f, \dot{\mathcal{P}}_u)$$

$$\therefore \quad \left| F(b) - F(a) - \int_a^b f \right| < \varepsilon, \quad \text{by } (*)$$

Since $\varepsilon > 0$ is arbitrary, $\int_a^b f = F(b) - F(a)$. $\#$

<u>Remarks</u>: (i) If $E = \phi$, then assumption (b) $\Rightarrow$ assumption (a).

(ii) One may allow f defined on $[a,b]$ except finite number of points as one can extend f to all $x \in [a,b]$ by setting $f(c) = 0$ for $c \notin$ domain(f) originally.

(iii) F differentiable on $[a,b] \not\Rightarrow F' \in \mathcal{R}[a,b]$

$\therefore$ assumption (c) is not automatically satisfied even $E = \phi$ & assumption (b) is satisfied. (Eg 7.3.2(e))

Eg 7.3.2

(a) • $F(x)=\frac{1}{2}x^2$, $\forall x\in[a,b]$ is continuous on $[a,b]$,

• $F'(x)=x$, $\forall x\in[a,b]$ ($\therefore E=\phi$)

• $F'(x)=x\in\mathcal{R}[a,b]$ (says by Thm 7.2.7, cts $\Rightarrow$ integrable)

$$\therefore \int_a^b x\,dx = F(b)-F(a)=\frac{1}{2}(b^2-a^2).$$

(b) Suppose $[a,b]$ is a closed interval s.t. ($\text{Arctan}\,x=\tan^{-1}x$)

$G(x)=\text{Arctan}\,x$ is defined on $[a,b]$ (for instance $[a,b]\subset(-\frac{\pi}{2},\frac{\pi}{2})$)

Then $G'(x)=\frac{1}{x^2+1}$, $\forall x\in[a,b]$ & is continuous on $[a,b]$

$\therefore$ (b) satisfied with $E=\phi$. (with $f(x)=\frac{1}{x^2+1}$)

Hence (a) satisfied automatically.

And Thm 7.2.7 $\Rightarrow$ (c) is also satisfied.

$$\therefore \int_a^b \frac{dx}{x^2+1} = \text{Arctan}\,b-\text{Arctan}\,a.$$

(c) $A(x) = |x|$ for $x \in [-10, 10]$. cts.

(one can do any $[\alpha, \beta]$ with $\alpha, \beta > 0$)

Then
$$A'(x) = \begin{cases} 1 & , \text{ for } x \in (0, 10] \\ \text{doesn't exist} & , \text{ for } x = 0 \\ -1 & , \text{ for } x \in [-10, 0) \end{cases}$$

Recall the signum function
$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$\therefore \quad A'(x) = \operatorname{sgn}(x) \quad \forall x \in [-10, 10] \setminus \{0\} \quad (E = \{0\})$

Note that $\operatorname{sgn}(x)$ is a step function, (or different from a step functions) at one point.

(with one degenerated interval)

Thm 7.2.5 $\Rightarrow \operatorname{sgn}(x) \in \mathcal{R}[-10, 10]$.

Hence $\displaystyle\int_{-10}^{10} \operatorname{sgn}(x)\,dx = A(10) - A(-10) = 10 - 10 = 0$.

(d) $H(x) = 2\sqrt{x}$ on $[0, b]$.

Then $H(x)$ cts on $[0, b]$,

$$H'(x) = \frac{1}{\sqrt{x}} \quad \forall x \in (0, b] \quad (E = \{0\})$$

Note that $h(x) = \frac{1}{\sqrt{x}}$ is unbounded on $[0,b]$,

$h \notin R[0,b]$ (No matter how we define $H'(0)$)

$\therefore$ Fundamental Thm 7.3.1 <u>doesn't</u> apply!

(Need to consider improper integrals, which is equivalent to applying Thm 7.3.1 to $[\varepsilon, b]$, and then letting $\varepsilon \to 0$.)

(e) $$K(x) = \begin{cases} x^2 \cos\left(\frac{1}{x^2}\right), & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

Then

$$K'(x) = \begin{cases} 2x\cos\frac{1}{x^2} + \frac{2}{x}\sin\left(\frac{1}{x^2}\right), & x \in (0,1] \\ 0, \text{ if } x = 0 & (\text{eg } 6.1.7(c)) \end{cases}$$

That is, K differentiable on $[0,1]$, & hence cts on $[0,1]$.

However K' is unbounded and

therefore $K' \notin R[0,1]$, assumption (c) doesn't satisfy!

Def 7.3.3: If $f \in \mathcal{R}[a,b]$, then the function defined by

$$F(z) = \int_a^z f \quad \text{for } z \in [a,b],$$

is called the indefinite integral of f with basepoint a.

(One may use other point as base point & is still called indefinite integral (Ex 7.3.6))

Thm 7.3.4 If $f \in \mathcal{R}[a,b]$, then

$$F(z) = \int_a^z f \quad \text{is continuous on } [a,b]$$

and in fact, if $|f(x)| \le M$, $\forall x \in [a,b]$, then

$$(*) \qquad |F(z) - F(w)| \le M|z-w|, \quad \forall z, w \in [a,b].$$

Remarks: (i) M exists because $f \in \mathcal{R}[a,b] \Rightarrow f$ is bdd

(ii) $(*)$ is called a Lipschitz condition, much stronger than just continuity.

Pf $\forall z, w \in [a,b]$ with $w \le z$, Additivity Thm 7.2.9 $\Rightarrow$

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f$$

$$\therefore \quad F(z) - F(w) = \int_w^z f.$$

If $\quad -M \le f(x) \le M, \quad \forall x \in [a,b],$

Thm 7.1.5 (c) $\Rightarrow$ $\quad -M(z-w) \leq \int_w^z f \leq M(z-w)$

$\therefore \quad |F(z)-F(w)| = \left|\int_w^z f\right| \leq M(z-w) = M|z-w|$

(since $w \leq z$)

Clearly, the case $z \leq w$ follows immediately too. #

Thm 7.3.5 (Fundamental Theorem of Calculus (2nd Form))

Let $f \in \mathcal{R}[a,b]$ and continuous at c.

Then $F(z) = \int_a^z f$ is differentiable at $z=c$ and

$$F'(c) = f(c).$$

Pf We'll prove only for the right-hand derivative

$$\lim_{h \to 0^+} \frac{F(c+h)-F(c)}{h} = f(c)$$

The left-hand derivative can be handled similarly.

Therefore, we assume $c \in [a,b)$.

Since f is continuous at c, $\forall \varepsilon > 0$, $\exists \eta_\varepsilon > 0$ s.t. if

(*) $\quad |f(x)-f(c)| < \varepsilon, \ \forall x \in [c, c+\eta_\varepsilon)$. (consider only right side)

Let $h \in (0, \eta_\varepsilon)$, then Additivity Thm 7.2.8 (Cor 7.2.10)

$\Rightarrow f \in \mathcal{R}[a, c+h]$, $\mathcal{R}[a,c]$ & $\mathcal{R}[c, c+h]$ and

$$\int_a^{c+h} f = \int_a^c f + \int_c^{c+h} f$$

i.e. $\quad F(c+h) - F(c) = \int_c^{c+h} f$

By $(*)$ $\quad f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \ \forall\, x \in [c, c+\eta_\varepsilon)$

we have $\quad (f(c)-\varepsilon)h \leq \int_c^{c+h} f \leq (f(c)+\varepsilon)h,$

which implies

$$f(c) - \varepsilon \leq \frac{F(c+h)-F(c)}{h} \leq f(c) + \varepsilon$$

$$\Rightarrow \quad \left| \frac{F(c+h)-F(c)}{h} - f(c) \right| \leq \varepsilon, \quad \forall\, h \in (0, \eta_\varepsilon)$$

It proves that $\quad \lim_{h \to 0^+} \frac{F(c+h)-F(c)}{h} = f(c)$

#

<u>Thm 7.3.6</u> If f is <u>continuous</u> on $[a,b]$, then

- $F(x) = \int_a^x f$ is <u>differentiable</u> on $[a,b]$, and
- $F'(x) = f(x), \ \forall\, x \in [a,b]$

Pf : f cts on $[a,b] \Rightarrow f \in \mathcal{R}[a,b]$ & cts at every pt. $c \in [a,b]$

#

Eg 7.3.7

(a) $f(x) = \operatorname{sgn} x$ on $[-1,1]$.

Then
- $f \in \mathcal{R}[-1,1]$ (step function with a degenerated interval)
- f not continuous at $x=0$, but continuous $\forall x \in [-1,1] \setminus \{0\}$.

Simply calculation: indefinite integral with basepoint -1 is

$$F(x) = \int_{-1}^{x} \operatorname{sgn}(x)\,dx = |x| - 1 \quad (\text{Ex!})$$

One can see that $F'(0)$ doesn't exist ("f cts at c" is a necessary condition)

and F is not an antiderivative of $f(x) = \operatorname{sgn}(x)$.

(b) Let h = Thomae's function

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1] \ \& \ \underset{\neq 0}{m}, n \text{ have no common factors } \ (\gcd(m,n)=1), \quad n \in \mathbb{N} = \{1,2,3,\cdots\} \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \text{ is irrational } \& \ x \in [0,1]. \end{cases}$$

Then by Eg 7.1.7, one concludes that

$$H(x) = \int_0^x h \equiv 0, \quad \forall x \in [0,1]$$

$$\Rightarrow \quad H'(x) = 0 \text{ exists } \forall x \in [0,1]$$

However, $H'(x) \neq h(x)$, $\forall$ rational $x \in [0,1]$.

Thm 7.3.8 (Substitution Theorem)

Let
- $f : I \to \mathbb{R}$ cts, (I = interval)
- $\varphi : [\alpha, \beta] \to \mathbb{R}$ st. $\varphi'(t)$ exists & cts $\forall t \in [\alpha, \beta]$.
  (i.e. φ has a continuous derivative)
- $\varphi([\alpha, \beta]) \subset I$

$$[\alpha,\beta] \xrightarrow{\varphi} I \xrightarrow{f} \mathbb{R}, \quad f \circ \varphi : [\alpha,\beta] \to \mathbb{R}$$

Then
$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

Notes: (i) t & x in the formula are dummy variables, just using them for convenient in practice:

thinking of change of variables $x = \varphi(t)$

In fact, the formula can be written as

$$\int_\alpha^\beta (f \circ \varphi) \cdot \varphi' = \int_{\varphi(\alpha)}^{\varphi(\beta)} f$$

(ii) The formula holds for $\varphi(\beta) \leq \varphi(\alpha)$ as we defined before.

Pf of Thm 7.3.8: Ex 7.3.17 (Easy application of Fundamental Thm & Chain rule)

Eg 7.3.9 Too easy, Omitted

Lebesgue's Integrability Criterion

Def 7.3.10

(a) A set $Z \subset \mathbb{R}$ is said to be a <u>null set</u> (<u>set of measure zero</u>)

if $\forall \varepsilon > 0$, $\exists$ a <u>countable</u> collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of

<u>open intervals</u> (could be overlapped) such that

$$Z \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \quad \text{and} \quad \sum_{k=1}^{\infty} (b_k - a_k) \leq \varepsilon$$

$\leftarrow$ length of interval (a_k, b_k)

(b) If $Q(x)$ is a <u>statement</u> about $x \in I$, we say that

" <u>$Q(x)$ holds almost everywhere on I</u> "

(or " $Q(x)$ holds for <u>almost every</u> (<u>almost all</u>) $x \in I$ ")

if $\exists$ a <u>null set</u> $Z \subset I$ s.t.

$$Q(x) \text{ holds } \forall x \in I \setminus Z.$$

In this case, we write $\quad Q(x)$ for a.e. $x \in I$.

<u>Remarks</u>: (i) "null set" may means "empty set" for some people.

So "set of measure zero" is used more often.

(ii) Def (a) means Z can be covered by a set of <u>arbitrary small</u> total length. (Kind of "length of $Z = 0$", but it is difficult to define "length" of arbitrary sets in $\mathbb{R}$.)

Eg 7.3.11 $\mathbb{Q}_1$ = set of rational numbers in $[0,1]$ is a null set.
(set of measure zero)

Pf: $\mathbb{Q}_1$ is countable and can be written as

$$\mathbb{Q}_1 = \{r_1, r_2, r_3, \cdots\}$$

Given $\varepsilon > 0$, define open intervals

$$J_k = \left(r_k - \frac{\varepsilon}{2^{k+1}}, r_k + \frac{\varepsilon}{2^{k+1}}\right), \quad k = 1, 2, \cdots$$

Clearly $r_k \in J_k$ and length of $J_k = \frac{\varepsilon}{2^k}$.

$\therefore$ $\mathbb{Q}_1 \subset \bigcup_{k=1}^{\infty} J_k$ and $\sum_{k=1}^{\infty}$ length of $J_k = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\mathbb{Q}_1$ is a null set.

From the proof, it is clear that it doesn't use the fact that r_k are rational. Hence, the proof can be used to prove that:

$$\boxed{\text{Every countable set is a null set (set of measure zero)}}$$

("countable infinite" can be proved similarly, "countable finite" are included by dropping the tail of the infinite series)

Thm 7.3.12 (Lebesgue's Integrability Criterion)

A bounded function $f: [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous almost everywhere on $[a,b]$

(Pf: Omitted. See App. C of the Textbook)

Eg 7.3.13

(a) Every step function on $[a,b]$ is bdd & has a finite set of points of discontinuity which is a set of measure zero and hence every step function on $[a,b]$ is Riemann integrable.

(b) Every monotone function on $[a,b]$ is Riemann integrable

In fact, monotone functions are bounded &

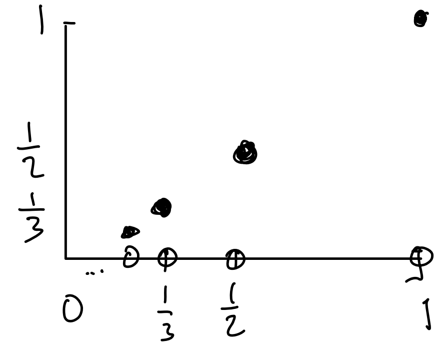
Thm 5.6.4 $\Rightarrow$ set of points of discontinuity of a monotonic function is countable.

Hence, it is a null set.

$\therefore$ Lebesgue's Integrability criterion $\Rightarrow$ it is Riemann integrable

(c) (eg 7.1.4(d))

$$G(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \quad (n=1,2,\cdots) \\ 0, & \text{elsewhere on } [0,1] \end{cases}$$



is bounded, and

Set of discontinuity $= \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

is countable hence measure zero.

Lebesgue's Integrability criterion $\Rightarrow$ $G(x)$ is Riemann integrable

(d) (Eg 7.2.2 (b), not integrable)

Dirichlet function $f(x) = \begin{cases} 1, & \text{if } x \text{ rational}, x \in [0,1] \\ 0, & \text{if } x \text{ irrational}, x \in [0,1]. \end{cases}$

is bounded.

set of discontinuity $= [0,1]$ (discontinuous at every $x \in [0,1]$)

which can be shown that it is <u>not a null set</u> (Omitted)

$\therefore$ Lebesgue's Integrability criterion $\Rightarrow$

Dirichlet function is not Riemann integrable.

(e) (eg 7.1.7) Thomae's function

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1] \text{ \& } m, n \text{ have no common factors} \\ & \phantom{\text{if }} n \in \mathbb{N} = \{1,2,3,\cdots\},\ m \neq 0 \quad (\gcd(m,n)=1) \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \text{ is irrational \& } x \in [0,1]. \end{cases}$$

is bounded & (by eg 5.1.6(h))

set of discontinuity = $\mathbb{Q}_1$ (set of rational numbers in $[0,1]$)

which is of measure zero (eg 7.3.11)

$\therefore$ Lebesgue's Integrability criterion $\Rightarrow$

Thomae's function is Riemann integrable on $[0,1]$.