<u>Thm 7.2.9</u> (<u>Additivity Theorem</u>)

Let $f:[a,b]\to\mathbb{R}$ & $c\in(a,b)$. $(a<b)$

Then $f\in\mathcal{R}[a,b] \iff f|_{[a,c]}\in\mathcal{R}[a,c]$ & $f|_{[c,b]}\in\mathcal{R}[c,b]$.

In this case $\int_a^b f = \int_a^c f + \int_c^b f$

<u>Pf</u> ($\Rightarrow$) By Cauchy Criterion (Thm 7.2.1)

$f\in\mathcal{R}[a,b]$

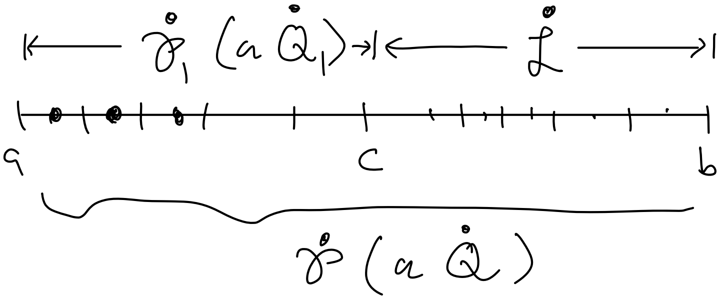
$\iff \forall\varepsilon>0,\ \exists\eta_\varepsilon>0$ s.t. $\forall\dot{\mathcal{P}},\dot{\mathcal{Q}}$ with $\|\dot{\mathcal{P}}\|<\eta_\varepsilon$ & $\|\dot{\mathcal{Q}}\|<\eta_\varepsilon$

we have $|S(f,\dot{\mathcal{P}})-S(f,\dot{\mathcal{Q}})|<\varepsilon$. —— $(*)_1$

Now we want to show that the same $\eta_\varepsilon>0$ works for the restriction $f_1=f_{[a,c]}:[a,c]\to\mathbb{R}$.

Suppose $\dot{\mathcal{P}}_1$ & $\dot{\mathcal{Q}}_1$ be two tagged partitions of $[a,c]$ with $\|\dot{\mathcal{P}}_1\|<\eta_\varepsilon$ & $\|\dot{\mathcal{Q}}_1\|<\eta_\varepsilon$.

Define now tagged partitions $\dot{\mathcal{P}}$ & $\dot{\mathcal{Q}}$ of $[a,b]$ by adding a tagged partition $\dot{\mathcal{L}}$ of $[c,b]$ with $\|\dot{\mathcal{L}}\|<\eta_\varepsilon$ to $\dot{\mathcal{P}}_1$ & $\dot{\mathcal{Q}}_1$



Then clearly $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ & $\|\dot{Q}\| < \eta_\varepsilon$

By $(*)_1$,

$$|S(f, \dot{\mathcal{P}}) - S(f, \dot{Q})| < \varepsilon .$$

On the other hand

$$S(f, \dot{\mathcal{P}}) = \underbrace{\sum_{x_i \le c} f(t_i)(x_i - x_{i-1})}_{\dot{\mathcal{P}}_1} + \underbrace{\sum_{x_{i-1} \ge c} f(t_i)(x_i - x_{i-1})}_{\dot{\mathcal{L}}}$$

and

$$S(f, \dot{Q}) = \underbrace{\sum_{x_i' \le c} f(t_i')(x_i' - x_{i-1}')}_{\dot{Q}_1} + \underbrace{\sum_{x_{i-1} \ge c} f(t_i)(x_i - x_{i-1})}_{\dot{\mathcal{L}}}$$

$$\therefore \quad S(f, \dot{\mathcal{P}}) - S(f, \dot{Q}) = S(f_1, \dot{\mathcal{P}}_1) - S(f_1, \dot{Q}_1)$$

$$\Rightarrow \qquad |S(f_1, \dot{\mathcal{P}}_1) - S(f_1, \dot{Q}_1)| < \varepsilon$$

Hence $f_1 : [a, c] \to \mathbb{R}$ satisfies Cauchy Criterion,

Therefore $f_1 \in \mathcal{R}[a, c]$.

Similarly, we have $\quad f_2 = f|_{[c,b]} \in \mathcal{R}[c, b]$.

$(\Leftarrow)$ Suppose $f_1 = f|_{[a,c]} \in \mathcal{R}[a,c]$ & $f_2 = f|_{[c,b]} \in \mathcal{R}[c,b]$.

Then Boundedness Thm 7.1.6 $\Rightarrow$ $f|_{[a,c]}$ & $f|_{[c,b]}$ are bdd.

$\Rightarrow$ f is bounded on $[a,b]$.

i.e. $\exists M > 0$ such that $|f(x)| \le M$, $\forall x \in [a,b]$.

Next let $L_1 = \int_a^c f_1 \left(= \int_a^c f\right)$ &

$L_2 = \int_c^b f_2 \left(= \int_c^b f\right)$

Then $\forall \varepsilon > 0$,

$\exists \delta' > 0$ s.t. $\forall$ tagged partition $\dot{\mathcal{P}}_1$ of $[a,c]$ with $\|\dot{\mathcal{P}}_1\| < \delta'$,

we have $|S(f_1, \dot{\mathcal{P}}_1) - L_1| < \varepsilon/3$

and

$\exists \delta'' > 0$ s.t. $\forall$ tagged partition $\dot{\mathcal{P}}_2$ of $[c,b]$ with $\|\dot{\mathcal{P}}_2\| < \delta''$,

we have $|S(f_2, \dot{\mathcal{P}}_2) - L_2| < \varepsilon/3$.

Now let $\delta_\varepsilon = \min\left\{\delta', \delta'', \frac{\varepsilon}{6M}\right\} > 0$ &

<u>Claim</u>: If $\dot{Q}$ is a tagged partition of $[a,b]$ with $\|\dot{Q}\| < \delta_\varepsilon$, then

$$|S(f; \dot{Q}) - (L_1 + L_2)| < \varepsilon.$$

If the Claim holds, then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = L_1 + L_2$ and we're done.

<u>Pf of claim</u>

Let $\dot{Q} = \{[x_{i-1}, x_i]; t_i\}_{i=1}^n$

then $x_i - x_{i-1} < \delta_\varepsilon$, $\forall i = 1, \cdots, n$.

<u>Case (i)</u> $c = x_k$ for some $k = 1, \cdots, n-1$. (excluding $x_0 = a$ & $x_n = b$)

Then $\dot{Q} = \{[x_{i-1}, x_i]; t_i\}_{i=1}^k \cup \{[x_{i-1}, x_i]; t_i\}_{i=k+1}^n$

Note that

$\dot{Q}_1 = \{[x_{i-1}, x_i]; t_i\}_{i=1}^k$ is a tagged partition of $[a, c]$ &

$\dot{Q}_2 = \{[x_{i-1}, x_i]; t_i\}_{i=k+1}^n$ is a tagged partition of $[c, b]$

Hence $S(f; \dot{Q}) = S(f_1; \dot{Q}_1) + S(f_2; \dot{Q}_2)$

Since $\|\dot{Q}_1\| \le \|\dot{Q}\| < \delta_\varepsilon \le \delta'$ &

$\|\dot{Q}_2\| \le \|\dot{Q}\| < \delta_\varepsilon \le \delta''$,

we have $|S(f_1; \dot{Q}_1) - L_1| < \varepsilon/3$

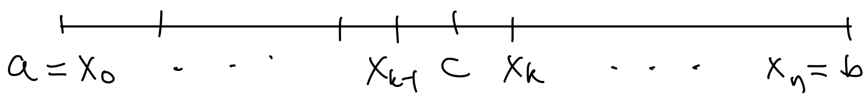
$|S(f_2; \dot{Q}_2) - L_2| < \varepsilon/3$.

Hence $\left| S(f;\dot{Q}) - (L_1+L_2) \right|$

$$\leq \left| S(f_1;\dot{Q}_1) - L_1 \right| + \left| S(f_2;\dot{Q}_2) - L_2 \right|$$

$$< \frac{2\varepsilon}{3} < \varepsilon .$$

<u>Case (ii)</u> $c \in (x_{k-1}, x_k)$ for some $k = 1, 2, \cdots, n$.



Then $[x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{k-2}, x_{k-1}] \cup [x_{k-1}, c]$

with tags $t_1, \quad t_2, \quad \cdots, \quad t_{k-1}, \quad \& \quad c$

is a tagged partition $\dot{Q}_1$ of $[a, c]$.

Similarly, $[c, x_k] \cup [x_k, x_{k+1}] \cup \cdots \cup [x_{n-1}, x_n]$ with tags

$c, \quad t_{k+1}, \quad \cdots, \quad t_n$

is a tagged partition $\dot{Q}_2$ of $[c, b]$.

Then

$S(f;\dot{Q})$

$$= \sum_{i=1}^{k-1} f(t_i)(x_i - x_{i-1}) + f(t_k)(x_k - x_{k-1}) + \sum_{i=k+1}^{n} f(t_i)(x_i - x_{i-1})$$

$$= \left[\sum_{i=1}^{k-1} f(t_i)(x_i - x_{i-1}) + f(c)(c - x_{k-1}) \right] - f(c)(c - x_{k-1})$$

$$+ f(t_k)(x_k - x_{k-1})$$

$$+ \left[f(c)(x_k - c) + \sum_{i=k+1}^{n} f(t_i)(x_i - x_{i-1}) \right] - f(c)(x_k - c)$$

$$= S(f_1, \mathring{Q}_1) - f(c)(c - x_{k-1}) + f(t_k)(x_k - x_{k-1})$$

$$+ S(f_2 ; \mathring{Q}_2) - f(c)(x_k - c)$$

$$\Rightarrow \quad |S(f; \mathring{Q}) - S(f_1, \mathring{Q}_1) - S(f_2; \mathring{Q}_2)|$$

$$\leq |f(t_k) - f(c)| \, |x_k - x_{k-1}|$$

$$\leq 2M \|\mathring{Q}\| < 2M \cdot \frac{\varepsilon}{6M}$$

$$< \frac{\varepsilon}{3} \qquad \text{———} \quad (*)_1$$

Also $\|\mathring{Q}_1\| \leq \|\mathring{Q}\|$ (as $0 < c - x_{k-1} < x_k - x_{k-1} \leq \|\mathring{Q}\|$)

$\therefore \quad \|\mathring{Q}_1\| < \delta_\varepsilon < \delta'$

$$\Rightarrow \quad | S(f_1, \mathring{Q}_1) - L_1 | < \varepsilon/3 . \qquad \text{———} \quad (*)_2$$

Similarly $\|\mathring{Q}_2\| \leq \|\mathring{Q}\| < \delta_\varepsilon < \delta''$

$$\Rightarrow \quad | S(f_2, \mathring{Q}_2) - L_2 | < \varepsilon/3 . \qquad \text{———} \quad (*)_3$$

Then by $(*)_1$, $(*)_2$, & $(*)_3$

$$\left| \tilde{S}(f;\mathring{Q}) - (L_1+L_2) \right|$$

$$\le \left| \tilde{S}(f;\mathring{Q}) - \tilde{S}(f_1;\mathring{Q}_1) - \tilde{S}(f_2;\mathring{Q}_2) \right|$$

$$+ \left| \tilde{S}(f_1;\mathring{Q}_1) - L_1 \right| + \left| \tilde{S}(f_2;\mathring{Q}_2) - L_2 \right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

This completes the proof of the Claim & hence the proof of the Thm. $\#$