Pf: (of Newton's Method)

Since $f(a)f(b)<0$, $f(a), f(b)$ have opposite signs (& nonzero)

f twice differentiable $\Rightarrow$ f cts. on $[a,b]$.

Intermediate Thm $\Rightarrow$ $\exists\, r\in(a,b)$ such that $f(r)=0$.

Note that $|f'(x)|\geq m>0$, $\forall x\in[a,b]$, Rolle's Thm

$\Rightarrow$ r is the unique zero of f in $[a,b]$.

i.e. $f(x)\neq 0$, $\forall x\in[a,b]\setminus\{r\}$, (Ex!)

Now $\forall x'\in I$, Taylor's Thm $\Rightarrow$

$$0=f(r)=f(x')+f'(x')(r-x')+\frac{f''(c')}{2}(r-x')^2$$

for some c' between r & x'.

(since f is twice diff.)

If $x''=x'-\dfrac{f(x')}{f'(x')}$, we have

$$x''=x'+\frac{f'(x')(r-x')+\frac{f''(c')}{2}(r-x')^2}{f'(x')}$$

$$=r+\frac{1}{2}\frac{f''(c')}{f'(x')}(r-x')^2$$

$$\Rightarrow\quad |x''-r|\leq\frac{1}{2}\frac{|f''(c')|}{|f'(x')|}|x'-r|^2$$

$$\leq\frac{1}{2}\frac{M}{m}|x'-r|^2=K|x'-r|^2. \qquad (*)$$

Choose $\delta > 0$ such that

$$\delta < 1/K \quad \& \quad [r-\delta, r+\delta] \subset [a,b],$$

and let $I^* = [r-\delta, r+\delta]$

Then, if $x_n \in I^*$ $(\subset [a,b])$ for some $n = 1, 2, 3, \cdots$,

we have, from $(*)$,

$$|x_{n+1} - r| \leq K\,|x_n - r|^2 \leq K\delta^2 < \delta$$

$$\therefore \quad x_{n+1} \in I^*.$$

i.e. $\quad x_n \in I^* \Rightarrow x_{n+1} \in I^*$.

Therefore, if $x_1 \in I^*$, induction $\Rightarrow$

the sequence $(x_n) \subset I^*$.

and satisfies the required inequality

$$|x_{n+1} - r| \leq K\,|x_n - r|^2, \quad \forall\, n = 1, 2, 3, \cdots$$

Finally, to see "limit", we note 1st that

$$|x_{n+1} - r| \leq K\,|x_n - r|^2 \leq K\delta\,|x_n - r| \qquad \text{——} \; (*)_2$$

Then iterate $(*)_2$:

$$|x_{n+1} - r| \leq (K\delta)\,|x_n - r| \leq (K\delta)\big(K\delta\,|x_{n-1} - r|\big)$$

$$= (K\delta)^2 |x_{n-1} - r| \leq \cdots$$

$$\leq (K\delta)^n |x_1 - r|$$

Since $K\delta < 1$, $(K\delta)^n \to 0$ as $n \to \infty$,

and $|x_1 - r|$ is a constant, we have

$$|x_{n+1} - r| \to 0 \quad \text{as } n \to \infty$$

i.e. $\qquad \lim_{n\to\infty} x_n = r$

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<u>eg 6.4.8</u> Using Newton's Method to approximate $\sqrt{2}$.

<u>Soln</u>: Convert the problem to a problem of finding root in order to use Newton's Method:

Consider $f(x) = x^2 - 2 \quad \forall x \in \mathbb{R}$.

Calculation: $\quad f'(x) = 2x \qquad$ ($\neq 0$ near the root, as 0 is not a root)

(f'' exists and satisfies the condition, but we don't need to find it explicitly in the approximation.)

One need to guess an <u>initial point</u> x_1.

Since $1^2 = 1$, $2^2 = 4$, $\quad (f(1) = -1,\ f(2) = 2)$

it seems reasonable to try $x_1 = 1$.

Note that $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 2}{2x_n}$$

$$= x_n - \frac{1}{2}x_n + \frac{1}{x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

$\therefore\ x_1 = 1 \Rightarrow$ $x_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5$

$x_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{17}{12} \simeq 1.416666$

$\vdots$

(Check!) $x_5 \approx 1.41421356237 2$ (correct to 11 places).

<u>Remarks</u>

(a) $(*)$ can be written as $\left(K|x_{n+1} - r|\right) \le \left(K|x_n - r|\right)^2$

Hence if $K|x_n - r| < 10^{-m}$,

then $K|x_{n+1} - r| < 10^{-2m}$

$\therefore$ number of significant digits in $K|x_n - r|$

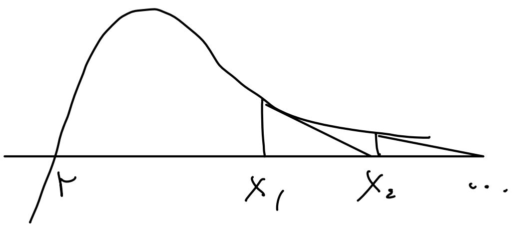
has been <u>doubled</u>.

And hence, the sequence (x_n) generated by Newton's method

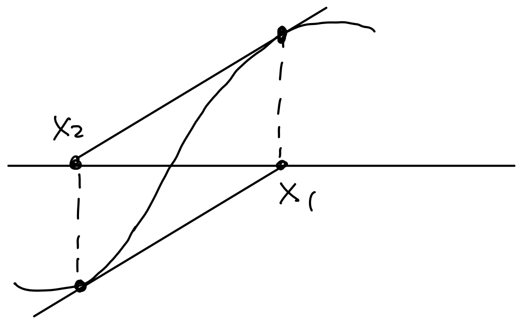
is said to "<u>converge quadratically</u>".

(b) Choose of initial x_1 is <u>important</u> (ie. has to be in I^*), otherwise (x_n) may not converge to the zero (root).

Possible situations



$(x_n \to \infty)$



(seq is $(x_1, x_2, x_1, x_2, x_1, x_2, \cdots)$)
no limit