§6.4 Taylor's Theorem

Recall: If $f(x)$ has n-th derivative at a point x_0, then the polynomial

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called the <u>n-th Taylor's Polynomial for f at x_0</u>.

Note: $P_n^{(k)}(x_0) = f^{(k)}(x_0) \quad \forall k = 0, 1, \cdots, n$.

<u>Thm 6.4.1</u> (<u>Taylor's Thm</u>)

Let
- $n \in \mathbb{N}$ (i.e. $n = 1, 2, \cdots$)
- $f : [a,b] \to \mathbb{R}$ such that $(a<b)$
- $f', \cdots, f^{(n)}$ are <u>continuous</u> on $[a,b]$ and
- $f^{(n+1)}$ exists on (a,b).

If $x_0 \in [a,b]$, then $\forall x \in [a,b]$, $\exists\, c$ <u>between x_0 and x</u> such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

where $P_n(x)$ is the n-th Taylor's Polynomial of f at x_0

__Remark__: $R_n(x) = f(x) - P_n(x)$ is referred as the __remainder__ and

$$R_n(x) = \frac{f^{(n+1)}(c)}{n+1}(x-x_0)^{n+1}$$

is called the __Lagrange form__ of the remainder,

or __derivative form__ of the remainder.

__Pf__ (of Thm 6.4.1)

Let $x_0, x \in [a,b]$ be given.

If $x_0 = x$, then the formula is clear.

If $x_0 \neq x$, we let

$J = [x_0, x]$ or $[x, x_0]$ depending on $x > x_0$ or $x_0 > x$.

Then J is a closed interval. $J \subset [a,b]$

Consider, for $t \in J$,

$$F(t) = f(x) - f(t) - (x-t)f'(t) - \frac{(x-t)^2}{2}f''(t) - \cdots$$
$$\cdots - \frac{(x-t)^n}{n!}f^{(n)}(t)$$

Then, • $F(x) = 0$,

• $F(x_0) = f(x) - P_n(x) = R_n(x)$ is the remainder

And, by assumption, • $F(t)$ is continuous on J, and

• $F'(t)$ exists in the <u>interior</u> of J

with
$$\begin{aligned} F'(t) = &- f'(t) \\ &+ f'(t) - (x-t)f''(t) \\ &+ (x-t)f''(t) - \frac{(x-t)^2}{2} f^{(3)}(t) \\ &+ \cdots \\ &+ \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) - \frac{(x-t)^n}{n!} f^{(n+1)}(t) \\ = &- \frac{(x-t)^n}{n!} f^{(n+1)}(t) . \end{aligned}$$

Consider further the function

$$G(t) = F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0) \quad \text{for } t \in J$$

Then G is continuous on J, differentiable in the <u>interior</u> of J,

and
$$\begin{cases} G(x_0) = F(x_0) - \left(\frac{x-x_0}{x-x_0}\right)^{n+1} F(x_0) = 0 \\ G(x) = F(x) - \left(\frac{x-x}{x-x_0}\right)^{n+1} F(x_0) = 0 \end{cases}$$

By Rolle's Thm, $\exists\, c \in$ interior of J (ie <u>between</u> x_0 & x)

s.t. $\quad 0 = G'(c) = F'(c) + (n+1) \dfrac{(x-c)^n}{(x-x_0)^{n+1}} F(x_0)$

$$\therefore \quad R_n(x) = F(x_0) = -\frac{1}{(n+1)} \cdot \frac{(x-x_0)^{n+1}}{(x-c)^n} F'(c)$$

$$= -\frac{1}{(n+1)} \cdot \frac{(x-x_0)^{n+1}}{(x-c)^n} \cdot \left(-\frac{(x-c)^n}{n!} f^{(n+1)}(c) \right)$$

$$= \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c) .$$

$\divideontimes$

Applications of Taylor's Theorem

eg 6.4.2 (Approximation of values)

(a) Use Taylor's Thm with $\underline{n=2}$ to approximate $\sqrt[3]{1+x}$, near $x=0$ $(x>-1)$.

Let $f(x) = (1+x)^{1/3}$, $x_0 = 0$

For $n=2$, $P_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$

using $f(x) = (1+x)^{1/3}$, $f(0) = 1$;

$\Rightarrow \quad f'(x) = \frac{1}{3}(1+x)^{-2/3}$, $f'(0) = \frac{1}{3}$

$\Rightarrow \quad f''(x) = -\frac{2}{9}(1+x)^{-\frac{5}{3}}$, $f''(0) = -\frac{2}{9}$

$\therefore \quad P_2(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2$

And hence $\quad f(x) = P_2(x) + R_2(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + R_2(x)$

where $R_2(x) = \frac{1}{3!} f'''(c)(x-x_0)^3 = \frac{1}{3!}\left(\frac{2\cdot 5}{9\cdot 3}\right)(1+c)^{-\frac{8}{3}} x^3$

$$= \frac{5}{81}(1+c)^{-\frac{8}{3}} x^3 \quad \text{for some } \underline{c \text{ between } 0 \,\&\, x}.$$

Explicit eg: If $x = 0.3$.

Then $P_2(0.3) = 1 + \frac{1}{3}(0.3) - \frac{1}{9}(0.3)^2 = 1.09$

$$R_2(0.3) = \frac{5}{81} \cdot \frac{1}{(1+c)^{8/3}} (0.3)^3$$

$\Rightarrow$ $|R_2(0.3)| \leq \frac{5}{81}(0.3)^3$ since $c \in (0, 0.3) \Rightarrow c > 0$ $\Rightarrow 1+c > 1$

$$= \frac{1}{600} < 0.0017$$

$\therefore$ $|f(0.3) - P_2(0.3)| < 0.0017$

i.e. $|\sqrt[3]{1.3} - 1.09| < 0.0017$

$\therefore$ $\sqrt[3]{1.3} \sim 1.09$ up to 2 decimal places.

(b) Use Taylor's Thm to approximate e with error $< 10^{-5}$ (5 decimal places)

(Assuming that we have defined e^x & proved $(e^x)' = e^x$, e^x increasing, & $e < 3$.)

Let $g(x) = e^x$, $x_0 = 0$.

Then by $(e^x)' = e^x$, we have $g^{(k)}(x) = e^x$, $\forall\, k = 1, 2, 3, \cdots$

Suppose that we need to use Taylor's Thm up to n.

Then the error is given by the remainder term

$$R_n(x) = \frac{1}{(n+1)!} e^c x^{n+1} \quad \text{for some } c \text{ between } 0 \text{ \& } x.$$

Take $x=1$, we have

$$R_n(1) \leq \frac{e}{(n+1)!} < \frac{3}{(n+1)!} \qquad (0<c<1)$$

Hence, to ensure error $< 10^{-5}$, we need

$$\frac{3}{(n+1)!} < 10^{-5}$$

i.e. $(n+1)! > 3 \cdot 10^5 = 300000$ (Should use the smallest possible n to reduce calculation)

Try: $(8+1)! = 9! = 362880$ $((7+1)! = 8! = 40{,}320)$

$\therefore$ $n=8$ is the required value and hence

$$e = g(1) \approx P_8(1) = g(0) + g'(0)\cdot 1 + \frac{g''(0)}{2!}\cdot 1^2 + \cdots + \frac{g^{(8)}(0)}{8!}\cdot 1^8$$

$$= 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{8!} \quad \text{with error} < 10^{-5}$$

$$= 2.718278\cdots \quad \text{(use calculator/computer)}$$

$\therefore$ $e = 2.71828$ upto 5 decimal places eg 6.4.3 (Applications to inequalities)

(a) $1-\frac{1}{2}x^2 \leq \cos x$, $\forall\, x \in \mathbb{R}$

Pf: Let $f(x)=\cos x$, $x_0=0$,

Then Taylor's Thm $\Rightarrow$

$$\cos x = 1-\frac{1}{2}x^2 + R_2(x) \qquad \text{(check!)}$$

with $$R_2(x) = \frac{f^{(3)}(c)}{3!}x^3 = \frac{\sin c}{6}x^3$$

for some c between 0 & x.

If $0 \leq x \leq \pi$, then $0 \leq c < \pi$ (the case $x=0$, we have $c=0$)

$\Rightarrow$ $\sin c \geq 0$, $x^3 \geq 0$

Hence $R_2(x) \geq 0$.

$\therefore$ $1-\frac{1}{2}x^2 \leq \cos x$ $\forall\, x \in [0,\pi]$.

If $x \in [-\pi, 0)$, then $y=-x \in (0,\pi]$

$$\Rightarrow 1-\frac{1}{2}y^2 \leq \cos y$$

Using $\cos(-x)=\cos x$, we have $1-\frac{1}{2}x^2 \leq \cos x$.

Hence $1-\frac{1}{2}x^2 \leq \cos x$, $\forall\, x \in [-\pi,\pi]$.

(check!)

If $|x|>\pi$, then $1-\frac{1}{2}x^2 < 1-\frac{1}{2}\pi^2 < -1 \leq \cos x$

All together $1-\frac{1}{2}x^2 \leq \cos x$ $\forall\, x \in \mathbb{R}$ #

(b) $\forall k=1,2,3,\cdots$ & $\forall x>0$

$$x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots-\frac{1}{2k}x^{2k} < \ln(1+x) < x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots+\frac{1}{2k+1}x^{2k+1}$$

Pf: Let $f(x)=\ln(1+x)$ for $x>-1$ (mistake in textbook)

Then $f'(x)=\frac{1}{1+x}$, $f''=\frac{-1}{(1+x)^2}$, $\cdots$ $f^{(n)}=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$\therefore$ $f^{(n)}(0)=(-1)^{n-1}(n-1)!$

$\Rightarrow$ nth Taylor's Poly of $\ln(1+x)$ at $x=0$ is

$$P_n(x)=0+1\cdot x-\frac{1}{2!}\cdot x^2+\frac{1}{3!}(2!)x^3-\cdots+\frac{1}{n!}(-1)^{n-1}(n-1)!x^n$$

$$=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots+\frac{(-1)^{n-1}}{n}x^n$$

and Remainder is

(mistake in Textbook)

$$R_n(x)=\frac{(-1)^n n!}{(n+1)!}\frac{1}{(1+c)^{n+1}}x^{n+1}$$ for some c between 0 and x

If $x>0$, then $c>0$, and hence $1+c>1$

$$\Rightarrow \quad R_n(x)=\frac{(-1)^n}{(n+1)}\cdot\left(\frac{x}{1+c}\right)^{n+1}\begin{cases} >0 & \text{if } n \text{ even} \\ <0 & \text{if } n \text{ odd.}\end{cases}$$

$\therefore$ For $n=2k$, $\ln(1+x)=P_{2k}(x)+R_{2k}(x)>P_{2k}(x)$

i.e. $\ln(1+x) > x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots-\frac{x^{2k}}{2k}$ $(\forall x>0)$

∴ For $n = 2k+1$ $\quad \ln(1+x) = P_{2k+1}(x) + R_{2k+1}(x) < P_{2k+1}(x)$

ie $\quad \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2k+1}}{2k+1} \qquad (\forall x > 0)$

※

(c) $e^{\pi} > \pi^{e}$

Pf: Taylor's Thm

$\Rightarrow \quad e^x = 1 + x + R_1(x) \qquad$ (see eg 6.4.2)

with $\quad R_1(x) = \frac{e^c}{2!} x^2 > 0$ for some c between 0 & x.

(using the fact that $e^c > 0, \ \forall c \in \mathbb{R}$)

$\therefore \quad e^x > 1 + x, \quad \forall x \neq 0$

Put $\quad x = \frac{\pi}{e} - 1 > 0 \quad$ (using known approx. values of π & e)

into it, we have

$$e^{(\frac{\pi}{e} - 1)} > 1 + \frac{\pi}{e} - 1 = \frac{\pi}{e}$$

$$\Rightarrow \quad e^{\frac{\pi}{e}} > \pi$$

$$\Rightarrow \quad e^{\pi} > \pi^{e}$$

※

Application to Relative Extrema (Higher Derivative Test)

Thm 6.4.4 Let
- $f: I \to \mathbb{R}$, (I = interval)
- x_0 be an interior point of I
- $f', f'', \cdots, f^{(n)}$ exist and continuous in a nbd of x_0.
- $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$

Then

(i) n even & $f^{(n)}(x_0) > 0 \implies f$ has a relative minimum at x_0

(ii) n even & $f^{(n)}(x_0) < 0 \implies f$ has a relative maximum at x_0

(iii) n odd $\implies f$ has neither a relative minimum nor a relative maximum at x_0

Remark: If $n=2$, it is the 2nd Derivative Test.

Pf: If $f^{(n)}(x_0) \neq 0$ and $f^{(n)}$ continuous,

then $\exists$ nbd $U = (x_0 - \delta, x_0 + \delta) \subset I$ of x_0 such that

$$\operatorname{sgn}(f^{(n)}(x)) = \operatorname{sgn}(f^{(n)}(x_0)), \quad \forall x \in U. \quad \text{— } (*)$$

Now, using $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$, the Taylor's Thm $\Rightarrow$

$$f(x) = f(x_0) + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-x_0)^n$$

$$= f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)^n, \quad \text{for some } c \text{ between } x_0 \text{ \& } x.$$

<u>Case (i)</u> n even, $f^{(n)}(x_0) > 0$.

By (*) & Taylor's, $\forall x \in U$

Since $\begin{cases} n \text{ even} \Rightarrow (x-x_0)^n \geq 0 \quad \forall x \in U \\ f^{(n)}(x_0) > 0 \Rightarrow f^n(c) > 0, \quad (x \in U \Rightarrow c \in U), \end{cases}$

$$f(x) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x-x_0)^n \geq 0,$$

$\therefore$ f has a relative minimum at x_0.

<u>Case (ii)</u> n even, $f^{(n)}(x_0) < 0$.

By (*) & Taylor's, $\forall x \in U$

Since $\begin{cases} n \text{ even} \Rightarrow (x-x_0)^n \geq 0 \quad \forall x \in U \\ f^{(n)}(x_0) < 0 \Rightarrow f^n(c) < 0, \quad (x \in U \Rightarrow c \in U) \end{cases}$

$$f(x) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x-x_0)^n \leq 0$$

$\therefore$ f has a relative maximum at x_0.

<u>Case (iii)</u> n odd

Taylor's Thm $\Rightarrow$ $\forall x \in U$

Since $\begin{cases} n \text{ odd} \Rightarrow (x-x_0)^n \text{ changes sign} \\ f^{(n)}(x_0) \neq 0 \Rightarrow f^{n}(c) \text{ has fixed sign } (x \in U \Rightarrow c \in U), \end{cases}$

$$f(x) - f(x_0) = \frac{f^{(n)}(c)}{n!}(x-x_0)^n \quad \text{changes sign}$$

$\therefore$ <u>Not</u> maximum and also <u>Not</u> minimum. #

Application to Convex Functions

Def 6.4.5 Let I be an interval.

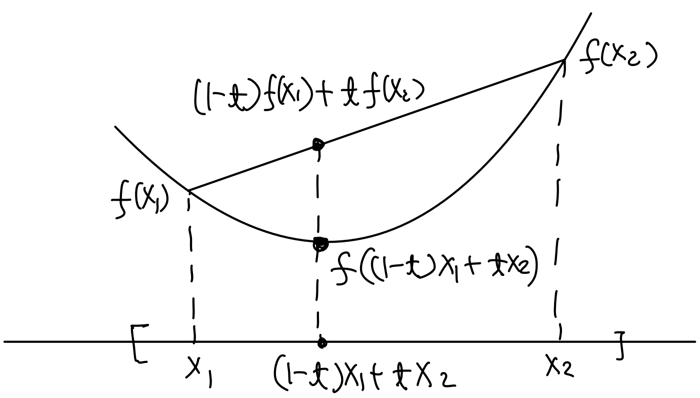
A function $f: I \to \mathbb{R}$ is said to be convex on I

if $\forall\, t \in [0,1]$ and any $x_1, x_2 \in I$, we have

$$f((1-t)x_1 + t x_2) \leq (1-t) f(x_1) + t f(x_2)$$

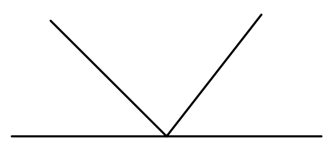
Geometric meaning:

Graph always below (at most up to) chord (with same end pts)



Remark: Convex function need not be differentiable

eg: $f(x) = |x|$ is convex



but not differentiable

Thm 6.4.6 Let
- $f: I \to \mathbb{R}$ (I (open) interval)
- $f''(x)$ exists $\forall\, x \in I$

Then f is convex on $I \iff f''(x) \geq 0,\ \forall\, x \in I$

Pf ($\Rightarrow$) (Ex 16 of §6.4)

$f''(x)$ exists $\Rightarrow$ $f''(x) = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

Now, f convex on I $\Rightarrow$

$\forall\, x \in I$ and $h \in \mathbb{R}$ such that $x \pm h \in I$, we have

$$f\left(\tfrac{1}{2}(x+h) + \tfrac{1}{2}(x-h)\right) \leq \tfrac{1}{2} f(x+h) + \tfrac{1}{2} f(x-h)$$

i.e.
$$2f(x) \leq f(x+h) + f(x-h)$$

Therefore
$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \geq 0$$

$$\Rightarrow \quad f''(x) = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \geq 0, \quad \forall\, x \in I.$$

($\Leftarrow$) Assuming $f''(x) \geq 0$, $\forall\, x \in I$.

$\forall\, t \in [0,1]$ & $\forall\, x_1, x_2 \in I$,

let $x_0 = (1-t)x_1 + t x_2$ (clearly $\in I$)

Then Taylor's Thm $\Rightarrow$

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \tfrac{1}{2} f''(c_1)(x_1 - x_0)^2$$

$$\geq f(x_0) + f'(x_0)(x_1 - x_0)$$

(for some c_1 between x_1 & x_0)

and $\quad f(x_2) = f(x_0) + f'(x_0)(x_2 - x_0) + \frac{1}{2} f''(c_2)(x_2 - x_0)^2$

(for some c_2 between x_2 & x_0)

$$\geq f(x_0) + f'(x_0)(x_2 - x_0)$$

$(1-t)f(x_1) + t f(x_2)$

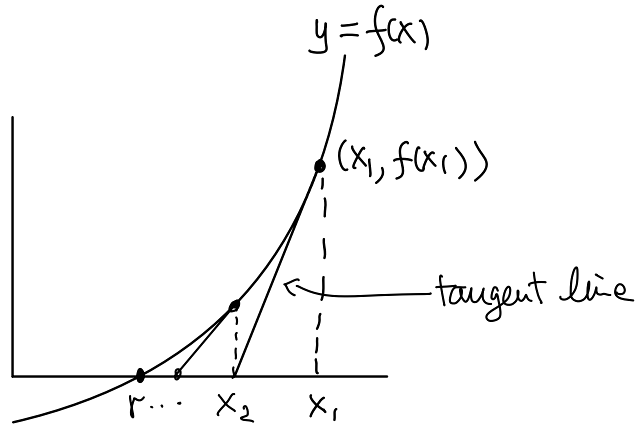
$$\geq (1-t)f(x_0) + t f(x_0) + f'(x_0)\left[(1-t)(x_1 - x_0) + t(x_2 - x_0)\right]$$

$$= f(x_0) + f'(x_0)\left[(1-t)x_1 + t x_2 - x_0\right]$$

$$= f((1-t)x_1 + t x_2), \quad \text{since} \quad x_0 = (1-t)x_1 + t x_2$$

Newton's Method

Idea:



$f(r) = 0$

Equation of tangent line: $\quad y - f(x_1) = f'(x_1)(x - x_1)$

$\therefore$ its intersection with x-axis, x_2, satisfies

$$-f(x_1) = f'(x_1)(x_2 - x_1)$$

$\therefore \quad x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} \quad$ (provided $f'(x_1) \neq 0$)

Successively $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \quad$ (provided $f'(x_k) \neq 0$ for $k = 1, \cdots, n$)

Hope (to find condition such that)

$$x_n \to r \quad (\text{a zero (root) of } f).$$

<u>Thm 6.4.7</u> <u>(Newton's Method</u>)

Let
- $f : [a,b] \to \mathbb{R}$ <u>twice</u> differentiable $\quad (a<b)$
- $f(a)f(b) < 0$ (ie $f(a)$, $f(b)$ have opposite signs)
- $\exists$ constants $m > 0$, $M \geq 0$ such that

$$|f'(x)| \geq m > 0 \quad \& \quad |f''(x)| \leq M, \qquad \forall x \in [a,b].$$

Then $\exists$ a <u>subinterval</u> $I^* \subset [a,b]$

- containing a zero r of f, such that
- $\forall x_1 \in I^*$, the sequence (x_n) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \forall n = 1, 2, 3 \cdots$$

belongs to I^* and

- $\lim_{n \to \infty} x_n = r$

Moreover
- $|x_{n+1} - r| \leq K |x_n - r|^2 \quad \forall n = 1, 2, 3 \cdots$

where $K = M/2m$.