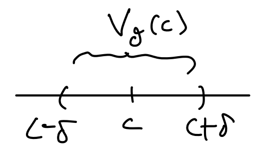
§6.2 The Mean Value Theorem

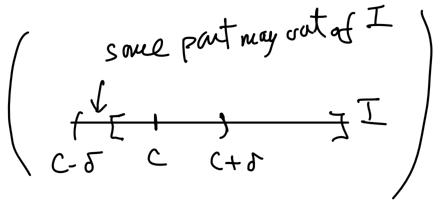
Recall: function $f: I \to \mathbb{R}$ is said to have a



- relative maximum at $c \in I$

if $\exists$ a neighborhood of c, $V = V_\delta(c) = (c-\delta, c+\delta)$ $(\delta > 0)$, such that

$$f(x) \le f(c), \quad \forall x \in V \cap I;$$



- relative minimum at $c \in I$

if $\exists$ a neighborhood of c, $V = V_\delta(c) = (c-\delta, c+\delta)$ $(\delta > 0)$, such that

$$f(x) \ge f(c), \quad \forall x \in V \cap I;$$

- relative extremum at $c \in I$ if either "relative maximum"

or "relative minimum"

Thm 6.2.1 (Interior Extremum Theorem) (Same notations as above)

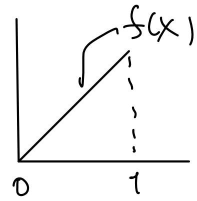
Let
- c be an interior point of the interval I
- f has a relative extremum at c.

If $f'(c)$ exists, then $f'(c) = 0$.

Note: The condition that $c \in I$ is an interior point is neccessary:

eg: $f(x)=x$ on $[0,1]$ has relative extremum

at $x=0$ (min), but $f'(0)=1\neq 0$.



(at $x=1$ (max), but $f'(1)=1\neq 0$.)

Pf (of Thm 6.2.1):

Prove only the case of relative maximum. The case of relative minimum is similar.

Let $c \in$ interior of I, f has a relative maximum at c and $f'(c)$ exists.

Suppose on the contrary that $f'(c)\neq 0$, then either

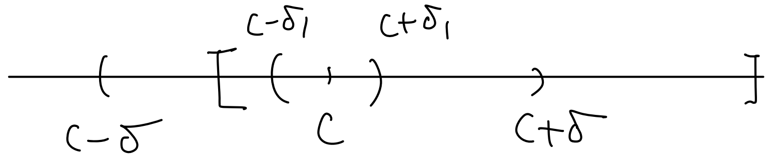
$$f'(c)>0 \quad \text{or} \quad f'(c)<0.$$

If $f'(c)>0$, i.e. $\displaystyle\lim_{\substack{x\to c\\(x\neq c)}} \frac{f(x)-f(c)}{x-c} > 0$.

Then (by Thm 4.2.9 of the Textbook, MATH2050), $\exists$ a nbd. $V=V_\delta(c)$

such that $\displaystyle\frac{f(x)-f(c)}{x-c} > 0 \quad \forall\ x\in V\cap I,\ x\neq c.$

Since $c \in$ interior of I, one can find a δ_1, $0<\delta_1<\delta$ (if needed) so that $(c-\delta_1, c+\delta_1) \subset V \cap I$.



Note that f has a relative maximum, there exists $\delta_2>0$ such that $f(x) \leq f(c)$, $\forall\, x \in (c-\delta_2, c+\delta_2) \cap I$

Then for $\delta_3 = \min\{\delta_1, \delta_2\} > 0$,

$$(c-\delta_3, c+\delta_3) \subset V \cap I \text{ and}$$

$$(c-\delta_3, c+\delta_3) \subset (c-\delta_2, c+\delta_2) \cap I$$

As a result,

$$\left.\begin{array}{l} \dfrac{f(x)-f(c)}{x-c} > 0, \\ \text{and } f(x) \leq f(c) \end{array}\right\} \quad \forall\, x \in (c-\delta_3, c+\delta_3),\ x \neq c.$$

Since $(c, c+\delta_3) \subset (c-\delta_3, c+\delta_3) \subset V \cap I$

The 1st inequality implies

$\exists\, x > c$, in $(c-\delta_3, c+\delta_3)$ s.t.

$$\frac{f(x)-f(c)}{x-c} > 0 \Rightarrow f(x)-f(c) > 0,$$

which contradicts the 2nd inequality.

Similarly, if $f'(c)<0$, one can find $\delta_3'>0$ so that

$$\left.\begin{aligned} &\frac{f(x)-f(c)}{x-c}<0, \\ \text{and}\quad &f(x)\le f(c) \end{aligned}\right\} \quad \forall x\in(c-\delta_3', c+\delta_3'),\ x\neq c.$$

The 1^{st} inequality

$\Rightarrow \exists\, x<c$ such that $\dfrac{f(x)-f(c)}{x-c}<0$.

$\Rightarrow \quad f(x)-f(c)>0$ contradicts the 2^{nd} inequality.

All together, we have

$$f'(c)=0. \quad \#$$

Cor 6.2.2 Let
- $f: I\to \mathbb{R}$ continuous
- f has a relative extremum at an interior point $c\in I$.

Then either

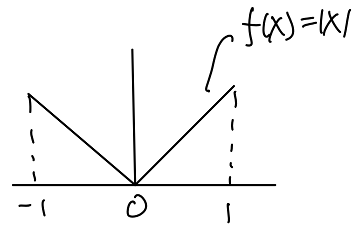
$$\begin{cases} \bullet\ f'(c) \text{ doesn't exist} \\ \text{or} \quad \bullet\ f'(c)=0. \end{cases}$$

(Pf: Follow easily from Thm 6.2.1)

eg: $f(x)=|x|$ on $I=[-1,1]$.

interior minimum at $x=0$.

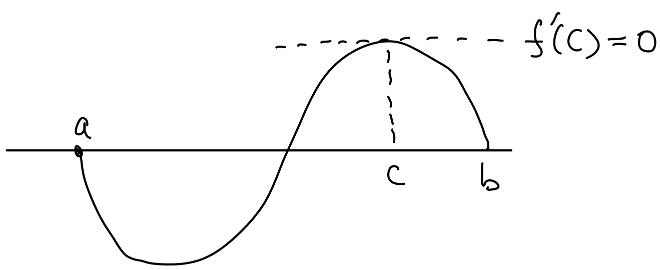
$f'(0)$ doesn't exist



Thm 6.2.3 (Rolle's Theorem)

Suppose
- $f:[a,b]\to\mathbb{R}$ continuous (on closed interval $I=[a,b]$) $(a<b)$
- $f'(x)$ exists $\forall\, x\in(a,b)$ (open interval, interior of I)
- $f(a)=f(b)=0$

Then $\exists\, c\in(a,b)$ such that $f'(c)=0$



Pf: If $f(x)\equiv 0$ on $[a,b]$, then $f'(x)=0$ $\forall\, x\in[a,b]$. We're done.

If $f(x)\not\equiv 0$, then either $f>0$ for some point in (a,b)

or $f<0$ for some point in (a,b).

Note that f is continuous on the closed interval $[a,b]$,

f attains an absolute maximum and an absolute minimum on I.

(Thm 5.3.4 of the Textbook, MATH2050)

Hence, if $f>0$ for some point in (a,b), f attains

the absolute maximum, ie. the value $\sup\{f(x): x\in I\}>0$,

at some point $c\in(a,b)$ as $f(a)=f(b)=0$.

Since $c\in(a,b)$, $f'(c)$ exists.

By Interior Extreme Theorem (Thm 6.2.1), $f'(c)=0$.

If there is no $x\in(a,b)$ s.t. $f>0$, then we must have

$f<0$ for some $x\in(a,b)$. Hence $(-f)>0$ for some $x\in(a,b)$

and $-f$ satisfies all conditions as f. Therefore,

$\exists\ c\in(a,b)$ such that $(-f)'(c)=0 \Rightarrow f'(c)=0$.

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Thm 6.2.4 (Mean Value Theorem)

Suppose
- $f:[a,b]\to\mathbb{R}$ continuous $\quad (a<b)$
- $f'(x)$ exists $\forall\, x\in(a,b)$

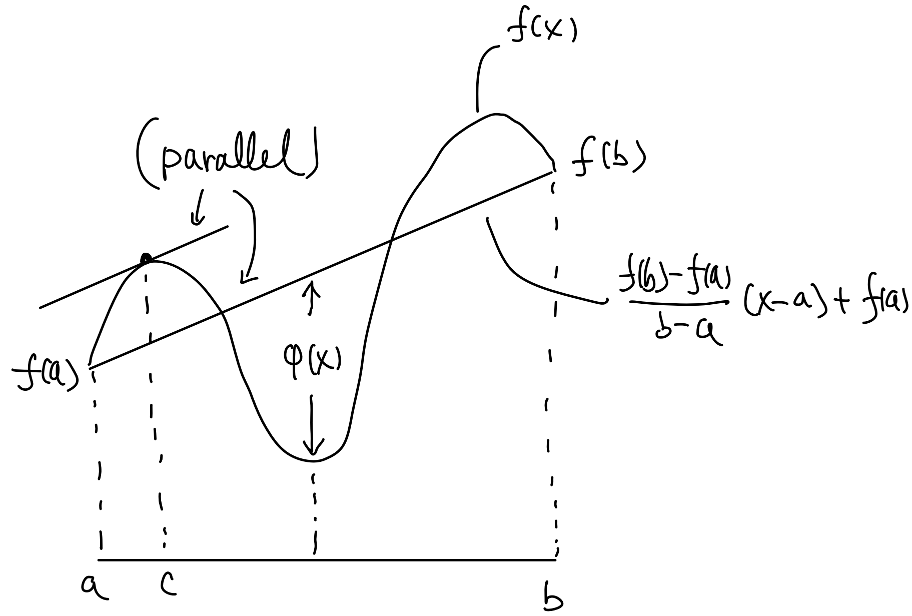
Then $\exists$ a point $c\in(a,b)$ such that

$$f(b)-f(a)=f'(c)(b-a)$$

Pf: Consider the function defined on $[a,b]$:

$$\varphi(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\right]$$

$$= f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$



Then φ is continuous on $[a,b]$ as f is continuous on $[a,b]$,

and $\varphi'(x)$ exists $\forall\, x\in(a,b)$ as $f'(x)$ exists $\forall\, x\in(a,b)$.

At the end points

$$\varphi(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0$$

$$\varphi(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = 0$$

$\therefore$ φ satisfies all conditions in Rolle's Thm (Thm 6.2.3).

Hence $\exists\ c \in (a,b)$ such that

$$0 = \varphi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} \qquad (\text{by Thm 6.1.3 and } (x)' = 1)$$

$$\therefore\ f(b) - f(a) = f'(c)(b-a). \quad ※$$

Applications of Mean Value Theorem

Thm 6.2.5 Suppose
- $f: [a,b] \to \mathbb{R}$ continuous $(a<b)$
- $f'(x)$ exists $\forall\ x \in (a,b)$ (ie. f differentiable on (a,b))
- $f'(x) = 0$, $\forall\ x \in (a,b)$.

Then f is a constant on $[a,b]$.

Pf: let $x \in [a,b]$ and $x > a$.

Applying Mean Value Thm to $f: [a,x] \to \mathbb{R}$,

(which clearly satisfies all conditions of the Thm) we find a point $c \in (a, x)$ such that

$$f(x) - f(a) = f'(c)(x-a) = 0 \quad (\text{by assumption } f'(c) = 0)$$

$$\Rightarrow \quad f(x) \equiv f(a), \ \forall x \in I.$$

$\therefore$ f is constant on I. ///

<u>Cor 6.2.6</u> Suppose • $f, g : [a,b] \to \mathbb{R}$ continuous

• f, g differentiable on (a,b)

• $f'(x) = g'(x), \ \forall x \in (a,b)$.

Then $\exists$ constant C such that $f = g + C$ on $[a,b]$.

<u>Recall</u> $f : I \to \mathbb{R}$ is said to be

- <u>increasing</u> on I if $x_1 < x_2 \ (x_1, x_2 \in I) \Rightarrow f(x_1) \leq f(x_2)$
  note: "not <"
- <u>decreasing</u> on I if $-f$ is increasing on I.

<u>Thm 6.2.7</u> Let $f : I \to \mathbb{R}$ be differentiable. Then

(a) f is increasing on $I \iff f'(x) \geq 0, \ \forall x \in I$

(b) f is decreasing on $I \iff f'(x) \leq 0, \ \forall x \in I$

Pf: (a) ($\Leftarrow$) Let $f'(x) \geq 0$, $\forall x \in I$.

Then for any $x_1, x_2 \in I$ with $x_1 < x_2$, we can apply

the Mean Value Thm to $f: [x_1, x_2] \to \mathbb{R}$

(since f is differentiable on $I \Rightarrow f: [x_1, x_2] \to \mathbb{R}$ satisfies all conditions of the MVT)

and find a point $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

$$\geq 0 \quad \text{since } f'(c) \geq 0 \text{ \& } x_2 > x_1.$$

$\therefore$ f is increasing on I.

(a) ($\Rightarrow$) Suppose f is differentiable and increasing on I.

Then $\forall c \in I$, we have

$$\frac{f(x) - f(c)}{x - c} \geq 0, \quad \forall x \in I, x \neq c$$

by "f is increasing" $\left(\begin{array}{l} \text{both "positive (or zero)" if } x > c, \\ \text{both "negative (or zero)" if } x < c \end{array} \right)$

Hence f differentiable at $c \Rightarrow$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0$$

(b) Applying (a) to $-f$. ※

Remarks: (1) Strictly increasing : $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

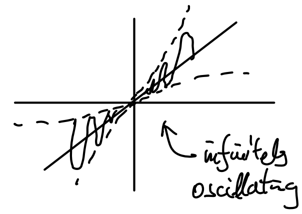
Then ex.13 of §6.2 $\Rightarrow$ " $f'(x) > 0$ on $I \Rightarrow f$ is strictly increasing on I ".

But : " $f'(x) > 0$ on I $\not\Leftarrow$ f is strictly increasing on I ".

Counterexample : $f(x) = x^3 : \mathbb{R} \to \mathbb{R}$ is strictly increasing,

but $f'(0) = 0$ which is not " > 0 ".

(2) Consider $g(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.



Exercise 10 of §6.2 : $g'(0) = 1 > 0$, but $g(x)$ is not increasing in any neighborhood of 0.

(That is, $g'(x) > 0$ only at a point x is not sufficient to ensure $g(x)$ is increasing near the point. We need a whole interval!)

Thm 6.2.8 (First Derivative Test for Extrema)

Let
- $f: [a,b] \to \mathbb{R}$ continuous $(a<b)$
- $c \in (a,b)$
- f is differentiable on (a,c) and (c,b).

Then (a) If $\exists\, \delta>0$ s.t.
$$\begin{cases} \bullet\ (c-\delta, c+\delta) \subseteq [a,b] \\ \bullet\ f'(x) \geq 0 \quad \text{for } x \in (c-\delta, c) \\ \bullet\ f'(x) \leq 0 \quad \text{for } x \in (c, c+\delta) \end{cases}$$

then f has a relative maximum at c.

(b) If $\exists\, \delta>0$ s.t.
$$\begin{cases} \bullet\ (c-\delta, c+\delta) \subseteq [a,b] \\ \bullet\ f'(x) \leq 0 \quad \text{for } x \in (c-\delta, c) \\ \bullet\ f'(x) \geq 0 \quad \text{for } x \in (c, c+\delta) \end{cases}$$

then f has a relative minimum at c.

Pf: (a) If $x \in (c-\delta, c)$, then Mean Value Thm

(applying to $f: [x,c] \to \mathbb{R}$) implies $\exists\, c_x \in (x,c)$ s.t.

$$f(c) - f(x) = f'(c_x)(c-x)$$
$$\geq 0 \quad (\text{since } f' \geq 0 \text{ on } (c-\delta, c))$$

If $x \in (c, c+\delta)$, then Mean Value Thm

(applying to $f: [c,x] \to \mathbb{R}$) implies $\exists\, c_x \in (c,x)$ s.t.

$$f(x) - f(c) = f'(c_x)(x-c)$$

$$\leq 0 \quad (\text{Since } f' \leq 0 \text{ on } (c, c+\delta))$$

Together we have $\quad f(c) \geq f(x) \quad \forall\, x \in (c-\delta, c+\delta)$

$\therefore$ f has a relative maximum at c.

(b) Applying (a) to $-f$. ※

<u>Remark</u>: <u>Converse of Thm 6.2.8</u> is not true.

i.e. $\exists$ differentiable function f has a relative maximum at c,

but the statement

"$\exists\, \delta > 0$ s.t. $\begin{cases} \bullet\ (c-\delta, c+\delta) \subseteq [a,b] \\ \bullet\ f'(x) \geq 0 \text{ for } x \in (c-\delta, c) \\ \bullet\ f'(x) \leq 0 \text{ for } x \in (c, c+\delta) \end{cases}$ "

is not true (Exercise 9 of §6.2)

Further Applications of the Mean Value Theorem

Examples 6.2.9

(a) Rolle's Thm 6.2.3 can be used to "locate" roots of a function.

In fact, Rollé's Thm $\Rightarrow$

$g = f'$ always has a root between any two zeros of f

(provided f is differentiable & etc.)

explicit eg: $\begin{cases} g(x) = \cos x = (\sin x)' \\ \sin x = 0 \text{ for } x = n\pi \text{ for } n \in \mathbb{Z}. \end{cases}$

Rolle's $\Rightarrow$ $\cos x$ has a root in $(n\pi, (n+1)\pi)$, $\forall n \in \mathbb{Z}$.

(eg. of Bessel functions J_n is omitted)

(b) Using Mean Value Theorem for approximate calculations & error estimates.

eg. Approximate $\sqrt{105}$.

Applying Mean Value Thm to $f(x) = \sqrt{x}$ on $[\overset{a}{\overset{\shortparallel}{100}}, \overset{b}{\overset{\shortparallel}{105}}]$,

$f(105) - f(100) = f'(c)(105 - 100)$ for some $c \in (100, 105)$.

In eg 6.1.10 (d), we've seen that $f'(c) = \frac{1}{2\sqrt{c}}$.

$\therefore$ $\sqrt{105} - \sqrt{100} = \frac{5}{2\sqrt{c}}$ for some $c \in (100, 105)$

$$\Rightarrow \quad 10+\frac{5}{2\sqrt{105}} < \sqrt{105} < 10+\frac{5}{2\sqrt{100}} = 10+\frac{5}{2\cdot 10} = 10.25$$

And $\sqrt{105} < \sqrt{121} = 11 \Rightarrow \sqrt{105} > 10 + \frac{5}{2\cdot 11}$

Hence $$\frac{205}{22} < \sqrt{105} < \frac{41}{4}$$

(Of course, the estimate can be improved by more careful analysis)

<u>Examples 6.2.10</u> (<u>Inequalities</u>)

(a) $e^x \geq 1+x$, $\forall\, x \in \mathbb{R}$ and "equality $\Leftrightarrow x=0$".

<u>Pf</u>: We will use the fact that

$f(x)=e^x$ has derivative $f'(x)=e^x$, $\forall x \in \mathbb{R}$

(and $f'(0)=1$)

and $e^x > 1$ for $x>0$

$e^x < 1$ for $x<0$.

(To be defined and proved in §8.3.)

If $x=0$, then $e^x = 1 = 1+x$. We're done.

If $x>0$, applying MVT (Mean Value Thm) to

$f(x)=e^x$ on $[0,x]$,

we have $c \in (0,x)$ such that

$$e^x - e^0 = e^c(x-0)$$

$$\therefore \quad e^x - 1 > x \, .$$

If $x<0$, applying MVT to $f(x)=e^x$ on $[x,0]$,

we have $c \in (x,0)$ such that

$$e^0 - e^x = e^c(0-x)$$

$$1-e^x < -x \qquad (e^c<1,\ -x>0)$$

$$\therefore \quad e^x > 1+x \ , \ \forall\ x<0.$$

Finally, one observes, in both cases, the inequality is strict.

So "equality $\Leftrightarrow$ $x=0$". ※

(b) $\quad -x \le \sin x \le x \ , \ \forall\ x \ge 0.$

Pf: The inequalities are clear for $x=0$.

Let $x>0$. Consider $g(x)=\sin x$ on $[0,x]$.

Then MVT implies $\exists\ c \in (0,x)$ s.t.

$$\sin x - \sin 0 = (\cos c)(x-0)$$

Using $-1 \le \cos c \le 1$ and $\sin 0 = 0$, we have

$$-x \le \sin x \le x \qquad (\text{as } x>0)$$ ※