9.2

4) Discuss the convergence or divergence of the series with n^{th} term

b) $n^n e^{-n}$: By the ratio test, we have

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{e^n}{e^{n+1}} = \frac{(n+1)^{n+1}}{n^n} \frac{1}{e} > 1 \text{ for } n \text{ large enough.}$$

Hence, the series diverges. ✓

d) $(\ln n)e^{-\sqrt{n}}$:

Since $\ln(n) < n$, we have

$(\ln n)e^{-\sqrt{n}} < ne^{-\sqrt{n}} = ne^{-n^{1/2}}$. We want to consider $ne^{-n^{\frac{1}{2}}}$ vs. $\frac{1}{n^2}$ for large n. So we have

$$\lim_{n\to\infty} \frac{ne^{-n^{\frac{1}{2}}}}{1/n^2} = \lim_{n\to\infty} \frac{n^3}{e^{\sqrt{n}}} \overset{LHR}{=} \lim_{n\to\infty} \frac{3n^2}{e^{\sqrt{n}}/2\sqrt{n}} = \lim_{n\to\infty} \frac{6n^{5/2}}{e^{\sqrt{n}}}$$

$$\overset{LHR}{=} \lim_{n\to\infty} \frac{15n^{\frac{3}{2}}}{e^{\sqrt{n}}/2\sqrt{n}} = \lim_{n\to\infty} \frac{30n^2}{e^{\sqrt{n}}} \overset{LHR}{=} \lim_{n\to\infty} \frac{60n}{e^{\sqrt{n}}/2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{120n^{\frac{3}{2}}}{e^{\sqrt{n}}} \overset{LHR}{=} \lim_{n\to\infty} \frac{180n^{\frac{1}{2}}}{e^{\sqrt{n}}/2\sqrt{n}} = \lim_{n\to\infty} \frac{360n}{e^{\sqrt{n}}}$$

$$\overset{LHR}{=} \lim_{n\to\infty} \frac{360}{e^{\sqrt{n}}/2\sqrt{n}} = \lim_{n\to\infty} \frac{720n^{\frac{1}{2}}}{e^{\sqrt{n}}} \overset{LHR}{=} \lim_{n\to\infty} \frac{720/\cancel{2\sqrt{n}}}{e^{\sqrt{n}}/\cancel{2\sqrt{n}}} = 0.$$

Hence for n large enough, we have

$(\ln n)e^{-\sqrt{n}} < ne^{-\sqrt{n}} < \frac{1}{n^2}$. So by comparison against $\sum \frac{1}{n^2}$, we have that $\sum (\ln n)e^{-\sqrt{n}}$ converges ✓

14) Show that the series

$$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\cdots \text{ is divergent.}$$

Pf: Note the $(3n)^{th}$ partial sum is given by

$$S_{3n} = 1+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\frac{1}{3n-2}+\left(\frac{1}{3n-1}-\frac{1}{3n}\right)$$

Then since forall n, $\frac{1}{3n-1} > \frac{1}{3n}$, we have

$$S_{3n} > 1+\frac{1}{4}+\frac{1}{7}+\cdots+\frac{1}{3n-2}$$

$$> \frac{1}{1+2}+\frac{1}{4+2}+\frac{1}{7+2}+\cdots+\frac{1}{3n}$$

$$= \frac{1}{3}\underbrace{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}$$

n^{th} partial sum of harmonic series, unbounded as $n\to\infty$, hence diverges.

9.3

1) Test the following series for convergence and for absolute convergence

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$:

By Cor 9.2.9, consider $\lim_{n\to\infty} \left(n\left(1-\left|\frac{a_{n+1}}{a_n}\right|\right)\right)$

$$= \lim_{n\to\infty}\left(n\left(1-\left|\frac{(-1)^{n+2}}{(n+1)^2+1}\cdot\frac{n^2+1}{(-1)^{n+1}}\right|\right)\right)$$

$$= \lim_{n\to\infty}\left(n\left(1-\left|(-1)\frac{n^2+1}{(n+1)^2+1}\right|\right)\right) = \lim_{n\to\infty}\left(n\left(1-\frac{n^2+1}{n^2+2n+2}\right)\right)$$

$$= \lim_{n\to\infty}\left(n-\frac{n^3+n}{n^2+2n+2}\right) = \lim_{n\to\infty}\frac{2n^2+n}{n^2+2n+2} = 2 > 1.$$

Hence, the series is absolutely convergent and therefore is also convergent. //

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$: Note that $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n+1}\right| = \sum_{n=1}^{\infty}\frac{1}{n+1}$ harmonic series and hence is not absolutely convergent.

Check non-absolute convergence:

By alternating series test, we have:

$\frac{1}{n+1} > 0$ for all n, $\lim_{n\to\infty}\frac{1}{n+1} = 0$, hence

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$ is convergent.

So $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$ is conditionally convergent. //

9) If the partial sums of $\sum a_n$ are bounded, show that the series $\sum_{n=1}^{\infty} a_n e^{-nt}$ converges for $t>0$.

Pf: Note that for $t>0$, $e^{-nt} \to 0$ as $n \to \infty$.

Then since the partial sums of $\sum a_n$ are bounded, we can apply Dirichlet's test (9.3.4) to see that

$\sum_{n=1}^{\infty} a_n e^{-nt}$ converges. //

9.4

6) b) $\sum_{n=0}^{\infty} \frac{n^\alpha}{n!} x^n$: $\alpha > 0$.

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{n^\alpha}{n!} \cdot \frac{(n+1)!}{(n+1)^\alpha} \right| = \lim_{n\to\infty} (n+1)\left(\frac{n}{n+1}\right)^\alpha$$

$$= \lim_{n\to\infty} \frac{n+1}{\left(1+\frac{1}{n}\right)^\alpha} = +\infty.$$

So radius of convergence $R = +\infty$.

d) $\sum_{n=2}^{\infty} \frac{x^n}{(\ln n)}$:

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{\ln(n+1)}{\ln(n)} \right| = 1$$

So radius convergence $R = 1$.