19/1/23 MATH 2060A Tutorial

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Sec 6.1

<u>Recall</u>: $f: \mathbb{R} \to \mathbb{R}$ is differentiable at c, if $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ s.t. if $x \in \mathbb{R}$ with $0 < |x - c| < \delta$, we have

w/ derivative L

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon \iff f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

$$\Downarrow$$

- Product Rule, Chain Rule, Power Rule, ...

$$|f(x) - f(c) - (x - c)L| < \varepsilon |x - c|$$

Q10: $f: \mathbb{R} \to \mathbb{R}$: $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$

Show that f is differentiable for all $x \in \mathbb{R}$, Also show that the derivative is unbounded on the interval $[-1, 1]$.

$x \neq 0$: $f'(x) = 2x \sin\left(\frac{1}{x^2}\right) + x^2 \cos\left(\frac{1}{x^2}\right)\left(-2x^{-3}\right)$

$= 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right).$

$x = 0$: $\lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x^2}\right)}{x} = \lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right) = 0.$

So f' exists for all $x \in \mathbb{R}$.

To show unboundedness, we'll show $\exists \{x_n\}_{n \geq 1}$ with $x_n \to 0$ as $n \to \infty$ but $|f'(x_n)| \to \infty$ as $n \to \infty$.

(For $N \in \mathbb{N}$ large enough, since $x_n \to 0$, $x_n \in [-1,1]$ for all $n \geq N$).

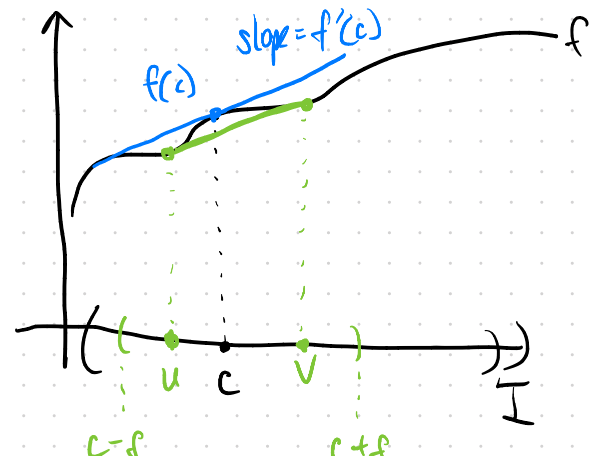
$$x_n := \frac{1}{\sqrt{2n\pi}} \to 0 \text{ as } n \to \infty.$$

$$|f'(x_n)| = \left| 2/\sqrt{2n\pi}\, \overset{0}{\cancel{\sin(2n\pi)}} - 2\sqrt{2n\pi}\, \overset{1}{\cancel{\cos(2n\pi)}} \right| = 2\sqrt{2n\pi} \to \infty \text{ as } n \to \infty.$$

Q17: Saddle lemma: Sps $f: I \to \mathbb{R}$ differentiable at $c \in I$, then show that $\forall \varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ s.t. if $u, v \in I$ with

$$c - \delta(\varepsilon) < u \leq c \leq v < c + \delta(\varepsilon), \text{ then}$$

$$|f(v) - f(u) - (v-u)f'(c)| \leq \varepsilon(v-u).$$



Pf: Take δ from def'n of derivative above.

$$|f(v) - f(c) - (v-c)f'(c) - (f(u) - f(c) - (c-u)f'(c))|$$

$\Delta \overset{\leq}{}$ $\leq |f(v)-f(c)-(v-c)f'(c)| + |f(u)-f(c)-(c-u)f'(c)|$

$\overset{\delta}{\leq} \varepsilon|v-c| + \varepsilon|c-u| = \varepsilon(v-u)$ //.

Sec 6.2

<u>Recall</u>: 1^{st} Derivative Test: Sps f: is cts at c and differentiable on some open nbhd of c, then if $\exists \delta>0$ s.t. for every $x \in (c-\delta, c)$, $f'(x) \leq 0$, and for every $x \in (c, c+\delta)$, $f'(x) \geq 0$, then f has a local minimum at c.

29: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 2x^4 + x^4 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$

Show that f has absolute minimum at 0, but that its derivative has both positive and negative values in every nbhd of 0.

(shows that converse to 1^{st} derivative test is not true).

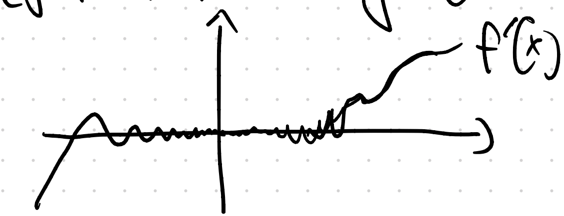
Pf: First we'll show for $x \neq 0$, $f(x) \geq 0$

Sps. $\exists x_0 \neq 0$ s.t. $f(x) < 0$, then $2x_0^4 + x_0^4 \sin\left(\frac{1}{x_0}\right) < 0$

$\iff 2x_0^4 < -x_0^4 \sin\left(\frac{1}{x_0}\right) \Rightarrow 2 < -\sin\left(\frac{1}{x_0}\right)$

contradicts $|\sin(y)| \leq 1$ for all y. $\lightning$

So 0 is a minimum of f.



$$f'(x) = \begin{cases} 8x^3 + 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

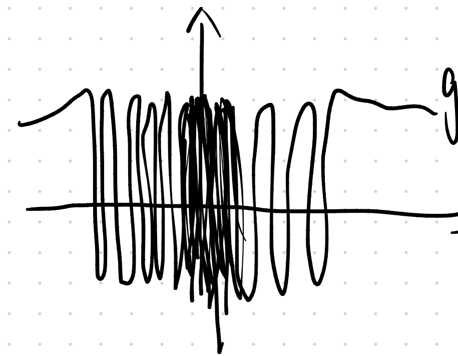
We'll show $\exists \{x_n\}, \{y_n\} \to 0$ as $n \to \infty$, but $f'(x_n) < 0$ for every x_n
$f'(y_n) > 0$ for every y_n.

(This is enough, $\forall \varepsilon > 0$, x_n, y_n will eventually be in $(-\varepsilon, \varepsilon)$ for n large enough)

$x_n := \frac{1}{2n\pi} \to 0$ as $n \to \infty$. $f'(x_n) = \frac{1}{n^3\pi^3} - 4n^2\pi^2 < 0$ for $n \geq 2$

$y_n := \frac{2}{(4n+1)\pi} \to 0$ as $n \to \infty$, $f'(y_n) = \frac{48}{(4n+1)^3 \pi^3} > 0$ for $n \geq 1$.

Q10: $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$



Show that g is not monotonic in any nbhd of 0.

Pf: $g'(x) = \begin{cases} 1 + 4x\sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 1, & x = 0 \end{cases}$

$x_n := \frac{1}{2n\pi} \to 0$ as $n \to \infty$, $g'(x_n) = \frac{1}{2n\pi} - 1 < 0$ for $n \geq 1$

$y_n := \frac{2}{(4n+1)\pi} \to 0$ as $n \to \infty$, $g'(y_n) = \frac{10}{(4n+1)\pi} > 0$ for $n \geq 1$.