THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 9
10th November2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. For each of the following function, if it is continuous on its domain, prove it by $\epsilon - \delta$ argument; otherwise find out its set of discontinuity (and prove it).
   (a) $f(x) := x^2$, where $x \in \mathbb{R}$.
   (b) $f(x) := \frac{x}{x^2-1}$ for $x \neq \pm 1$, $f(x) := 1$ for $x = 1$ and $f(x) := -1$ for $x = -1$.
   (c) For $x \in [0, 1]$, if x is irrational, define $f(x) := x$, and if x is rational, write it in reduced fraction form $x = p/q$ where $\gcd(p, q) = 1$, and define $f(x) = p\sin(1/q)$. (Hint: you are allowed to use the fact that $\lim_{n\to\infty} n \sin(1/n) = 1$.)
2. Suppose that f, g are functions on $A \subset \mathbb{R}$, if f is continuous and g is discontinuous on A, is $f + g$ necessarily discontinuous?
3. Give an example of a pair of discontinuous functions f, g on $\mathbb{R}$ so that their composition $g \circ f$ is continuous.
4. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function, i.e. $f(x + y) = f(x) + f(y)$ for any x, y, show that if f is continuous at some point c, then f is continuous on $\mathbb{R}$.
5. Let $g : \mathbb{R} \to \mathbb{R}$ be multiplicative, i.e. $g(x + y) = g(x)g(y)$ for any x, y, show that if g is continuous at 0, then g is continuous on $\mathbb{R}$.
6. In the following exercise, we will prove that the set of discontinuity D_f of a function $f : \mathbb{R} \to \mathbb{R}$ is a countable union of closed subsets (such are called F_σ sets).
   (a) For each $\epsilon > 0$, we say that f is ϵ-continuous at c if there exists $\delta > 0$ so that for all x, y in $(c - \delta, c + \delta)$, we have $|f(x) - f(y)| < \epsilon$. We denote $D_\epsilon = \{c \in \mathbb{R}|\ f \text{ is not } \epsilon\text{-continuous at } c\}$. Prove that D_ϵ is a closed subset for any ϵ.
   (b) Show that for $\epsilon_1 < \epsilon_2$, we have $D_{\epsilon_2} \subset D_{\epsilon_1}$.
   (c) Prove that for any $\epsilon > 0$, if f is continuous at c, then f is ϵ-continuous at c. Hence deduce that $D_\epsilon \subset D_f$.
   (d) Prove that if f is not continuous at c, then there is some ϵ so that f is not ϵ-continuous. Hence show that $D_f = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}$.
   Remark: Actually, the converse is also true. Given any F_σ set, it is the set of discontinuity of some function on $\mathbb{R}$.