THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 3 solutions
29th September 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. (a) We claim that $\lim \frac{3n+1}{2n+5} = \frac{3}{2}$. To see this, given any $\epsilon > 0$, consider

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{6n+2-6n-15}{4n+10}\right| = \frac{13}{4n+10} < \frac{13}{4n}.$$

Therefore, if we pick $N_0 \in \mathbb{N}$ so that $N_0 > \frac{13}{4\epsilon}$, whose existence is guaranteed by Archimedean property. Then for any $n \geq N_0$, according to the above,

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \frac{13}{4n} \leq \frac{13}{4N_0} < \epsilon.$$

(b) We claim that $\lim \frac{n^2-1}{2n^2+3} = \frac{1}{2}$. Given any $\epsilon > 0$, consider

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{2n^2-2-2n^2-3}{4n^2+6}\right| = \frac{5}{4n^2+6} < \frac{5}{4n^2}.$$

If we pick $N_0 \in \mathbb{N}$ by AP so that $N_0 > \sqrt{\frac{5}{4\epsilon}}$, then for any $n \geq N_0$, we have

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| < \frac{5}{4n^2} \leq \frac{5}{4N_0^2} < \epsilon.$$

(c) We claim that $\lim \sqrt{4n^2+n} - 2n = \frac{1}{4}$. Given any $\epsilon > 0$, we note that

$$\left|\sqrt{4n^2+n} - 2n - \frac{1}{4}\right| = \left|\frac{(4n^2+n) - (2n+\frac{1}{4})^2}{\sqrt{4n^2+n} + 2n + \frac{1}{4}}\right| = \frac{1/16}{\sqrt{4n^2+n} + 2n + 1/4}.$$

The latter expression is simply less than $\frac{1}{32n}$. Therefore if we pick $N_0 \in \mathbb{N}$ by AP so that $N_0 > \frac{1}{32\epsilon}$, then for any $n \geq N_0$, we have

$$\left|\sqrt{4n^2+n} - 2n - \frac{1}{4}\right| < \frac{1}{32n} \leq \frac{1}{32N_0} < \epsilon.$$

(d) We will apply Bernoulli's inequality to show that $\lim na^n = 0$ for $0 < a < 1$. First rewrite $a = 1/(1+r)$ where $r > 0$, then by $(1+r)^n = 1 + nr + \frac{n(n-1)}{2}r^2 + ... \geq \frac{n(n-1)}{2}r^2$, we have for $n \geq 2$,

$$na^n = \frac{n}{(1+r)^n} \leq \frac{n}{n(n-1)r^2/2} = \frac{2}{(n-1)r^2}.$$

So given any $\epsilon > 0$, we can choose $N_0 \in \mathbb{N}$ so that $N_0 \geq 2$ and $N_0 > \frac{r^2}{2\epsilon} + 1$. Then for any $n \geq N_0$, we have

$$na^n \leq \frac{2}{(n-1)r^2} \leq \frac{2}{(N_0-1)r^2} < \epsilon.$$

(e) (Method 1) Let $b > 1$, then by monotonicity of n-th root, $b^{\frac{1}{n}} > 1^{\frac{1}{n}} = 1$. So we can consider $y_n := b^{\frac{1}{n}} - 1 > 0$. Then by Bernoulli's inequality,

$$b = (1+y_n)^n \geq 1 + ny_n > ny_n.$$

So we have $b/n > y_n > 0$. Given $\epsilon > 0$, we choose $N_0 \in \mathbb{N}$ so that $N_0 \geq b/\epsilon$, then for $n \geq N_0$, we have

$$y_n = |b^{\frac{1}{n}} - 1| < \frac{b}{n} \leq \frac{b}{N_0} < \epsilon.$$

(Method 2) Write $b = 1 + r$, the claim is that $(1+r)^{\frac{1}{n}} \leq 1 + \frac{r}{n}$. To see this, simply take n-th power of both sides, we get $1 + r \leq 1 + n \cdot \frac{r}{n} + \frac{n(n-1)}{2}\frac{r^2}{n^2} + ...$ which is clearly true. Since taking n-th power is order preserving, we obtain the first inequality. Then given any $\epsilon > 0$, we can pick $N_0 \in \mathbb{N}$ so that $N_0 > \frac{r}{\epsilon}$. Then for $n \geq N_0$, we have

$$|b^{\frac{1}{n}} - 1| \leq \frac{r}{n} \leq \frac{r}{N_0} < \epsilon.$$

(f) By taking $c = \frac{1}{b}$, then $b > 1$ and we may apply part (e) to conclude that $c^{\frac{1}{n}} = \frac{1}{b^{\frac{1}{n}}} \to 1/1 = 1$.

(g) Again writing $x_n = n^{\frac{1}{n}} = 1 + y_n$, note that $y_n > 0$ and then $n = x_n^n = (1+y_n)^n \geq 1 + ny_n + n(n-1)y_n^2/2 > n(n-1)y_n^2/2$. Therefore we have the inequality when $n > 1$,

$$\sqrt{\frac{2}{n-1}} \geq y_n \geq 0.$$

So given $\epsilon > 0$, we may pick $N_0 \in \mathbb{N}$ so that $N_0 \geq 2$ and $N_0 > 1 + \frac{2}{\epsilon^2}$. Then for any $n \geq N_0$, we have

$$|n^{\frac{1}{n}} - 1| = y_n \leq \sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{2}{N_0-1}} < \epsilon.$$

2. We will prove that (x_n) is convergent using bounded monotone theorem. First, we show that if $x_n \geq \sqrt{2}$ then so is $x_{n+1} \geq \sqrt{2}$. This is direct since $x_{n+1} = \frac{1}{2}(x_n + 2/x_n) \geq \frac{1}{2}(\sqrt{2} + 2/\sqrt{2}) = \sqrt{2}$, there (x_n) is bounded below by $\sqrt{2}$. Next, we note that x_n is monotonic decreasing, as

$$x_{n+1} - x_n = \frac{1}{2}\left(\frac{2}{x_n} - x_n\right) \leq \frac{1}{2}\left(\frac{2}{\sqrt{2}} - \sqrt{2}\right) = 0.$$

Hence (x_n) is convergent. Denote $L = \lim x_n$, then again $L = \lim x_{n+1}$ and the recursively relation implies that

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right).$$

Therefore $L^2 = 2$ and hence $L = \sqrt{2}$. Note that L cannot be $-\sqrt{2}$ because otherwise, there is some $x_j < -\sqrt{2} + \sqrt{2} = 0 < \sqrt{2}$, contradicting the fact that x_n are bounded below by $\sqrt{2}$.

3. Suppose that $\lim \frac{x_{n+1}}{x_n} = c < 1$, then pick $\epsilon_0 > 0$ small enough so that $q := c + \epsilon_0 < 1$ still. Then there exists some $N_0 \in \mathbb{N}$ so that for $n \geq N_0$, we have

$$\frac{x_{n+1}}{x_n} - c < \epsilon.$$

Therefore for $n > N_0$, we have

$$0 < x_n = x_{N_0}\frac{x_{N_0+1}}{x_{N_0}} \cdots \frac{x_n}{x_{n-1}} < x_{N_0}q^{n-N_0+1}.$$

By Tutorial 2 Q6, the RHS of the above has limit equals to 0. Therefore by squeeze theorem, we have $\lim x_n = 0$.

4. No, the harmonic series $x_n := \sum_{k=1}^{n} \frac{1}{k}$ provides a counter example, clearly $|x_{n+1} - x_n| = \frac{1}{n+1} < \frac{1}{n}$. It is a divergent sequence because it is unbounded. Given any $0 < M \in \mathbb{N}$, we have

$$x_{2^M} = \sum_{k=1}^{2^M} \frac{1}{k} > \frac{1}{2} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + ... = \frac{M+1}{2}.$$

Here, we are bounded $\frac{1}{k} > \frac{1}{2^j}$ for $2^{j-1} + 1 < k < 2^j$, therefore

$$\sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} > \frac{2^{j-1}}{2^j} = \frac{1}{2}.$$

Since M is arbitrary, for any real number, there is some x_n greater than the chosen real number. By proposition 2.7, x_n cannot be convergent.

5. Note that $x \mapsto x^{\frac{1}{n}}$ is an order preserving function. Therefore taking n-th root on the inequality given in the assumption yields

$$\delta^{\frac{1}{n}} < x_n^{\frac{1}{n}} < n^{\frac{k}{n}}.$$

According to Q1 part e,f and g, we know that $\lim \delta^{\frac{1}{n}} = \lim n^{\frac{1}{n}} = 1$. Then by squeeze theorem, $\lim x_n^{\frac{1}{n}} = 1$ as well.

6. Suppose that $\lim x_n = L$, then by triangle inequality,

$$\left|\frac{x_1 + ... + x_n}{n} - L\right| = \left|\frac{x_1 - L}{n} + ... + \frac{x_n - L}{n}\right| \leq \left|\frac{x_1 - L}{n}\right| + ... + \left|\frac{x_n - L}{n}\right|.$$

Given $\epsilon > 0$, we can find some $N_0 \in \mathbb{N}$ so that for $m > N_0$ $|x_m - L| < \epsilon$. For this choice of N_0, write $M = \sum_{i=1}^{N_0} |x_i - L|$. We can find some other $N_1 \in \mathbb{N}$ so that $\frac{M}{n} < \epsilon$ for $n \geq N_1$.

Then for $n > \max N_0, N_1$, we have

$$\begin{aligned}\left|\frac{x_1 + ... + x_n}{n} - L\right| &\leq \frac{1}{n}\sum_{j=1}^{N_0} |x_j - L| + \frac{1}{n}\sum_{j=N_0+1}^{n} |x_j - L| \\ &\leq \frac{M}{n} + \frac{n - N_0}{n}\epsilon \\ &< \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

This proves the convergence of y_n.

For a counter-example of the converse of the statement. Consider $x_n = (-1)^{n+1}$, then it is divergent since given any L, L will have distance greater than 1 with 1 or -1, so if we pick $\epsilon = 1$, we see that $\lim x_n \neq L$. However, $y_n = \frac{1}{n}(x_1 + ... + x_n) = \frac{1}{n}$ when n is odd, and $y_n = 0$ when n is even. Then it is clear that $\lim y_n = 0$.