THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 1
15th September 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. For each of the following subsets, determine whether its supremum and infimum exist. If they exist, find their values.
   (a) $X = \{q^2 | \, q \in \mathbb{Q}\}$.
   (b) $X = \{(-1)^n/n | \, n \in \mathbb{N}\}$.
   (c) $X = \{x \in \mathbb{R} | \, 4x - x^2 > 3\}$.
   (d) Let $r \in \mathbb{R}$ be arbitrary, $X = \{|q - r| : \, q \in \mathbb{Q}\}$.
2. Show that for a subset A, if $\sup(A)$ exists, then so does $\inf(-A)$. Prove the formula $\inf(-A) = -\sup(A)$.
3. For two subsets A, B, define $A - B = \{a - b | \, a \in A, b \in B\}$. Show that $\inf(A - B) = \inf A - \sup B$.
4. Let A and B be two nonempty subsets of $\mathbb{R}$ that satisfy the property that, for any pair of $a \in A$ and $b \in B$, $a \leq b$. Prove that $\sup A \leq \inf B$.
5. Let T be a bounded subset in $\mathbb{R}$ and $S \subset T$ a nonempty subset, show that $\inf T \leq \inf S \leq \sup S \leq \sup T$.
6. Suppose that S is a nonempty subset that contains an upper bound of itself, show that such element is unique and it must be the supremum.
7. According to lecture note P.3, it is possible to deduce $\inf\{1/n | \, n \in \mathbb{N}\} = 0$ from Archimedean property of $\mathbb{R}$. Prove the converse statement, i.e., assume that we know $\inf\{1/n | \, n \in \mathbb{N}\} = 0$, without invoking completeness axiom, prove the Archimedean property.
8. Given $D \subset \mathbb{R}$, and a function $f : D \to \mathbb{R}$, we denote

$$f(D) = \{y \in \mathbb{R} | \, y = f(x) \text{ for some } x \in D\}$$

and

$$\sup_{x \in D} f(x) = \sup f(D).$$

Now suppose $f, g : D \to \mathbb{R}$ are functions on the same domain D.

(a) If $f(x) \leq g(x)$ for all $x \in D$, show that $\sup_{x\in D} f(x) \leq \sup_{x\in D} g(x)$.

(b) If $f(x) \leq g(y)$ for all $x, y \in D$, show that $\sup_{x\in D} f(x) \leq \inf_{x\in D} g(x)$.

(c) Find an example in which the premise of part (a) holds but the conclusion of part (b) does not.

9. Let $D = (0,1) = \{0 < x < 1 | \, x \in \mathbb{R}\}$, consider the function $h : (0,1)^2 \to \mathbb{R}$ defined by $h(x,y) = 2x + y$.

   (a) For each $x \in D$, compute $f(x) := \sup_{y\in D} h(x,y)$ in terms of x, then find the value $\inf_{x\in D} f(x)$.

   (b) For each $y \in D$, compute $g(y) := \inf_{x\in D} h(x,y)$ in terms of y, then find the value $\sup_{y\in D} g(y)$.

10. Repeat exercise Q7 for the function $h : (0,1)^2 \to \mathbb{R}$, where $h(x,y) = 0$ for $x < y$ and $h(x,y) = 1$ for $x \geq y$.

11. Prove that for X, Y subsets of $\mathbb{R}$, and $h : X \times Y \to \mathbb{R}$ be a function, then

$$\sup_{y\in Y} \inf_{x\in X} h(x,y) \leq \inf_{x\in X} \sup_{y\in Y} h(x,y).$$

    Note that in particular, Q7 and Q8 demonstrate the above inequality can be either an equality or a strict inequality.