THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Test 1 solutions
21st October 2022

- Please send an email to echlam@math.cuhk.edu.hk if you have any questions.

1. (20 points)

   For a non-negative function q on $\mathbb{R}$, a real sequence (x_n) is called q-Cauchy if for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ so that for $m, n \geq N$, we have $|q(x_m - x_n)| < \epsilon$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function satisfying (i) $\phi(s+t) = \phi(s) + \phi(t)$ for any $s, t \in \mathbb{R}$, and (ii) for any (x_n) so that $x := \lim x_n$ and $y := \lim \phi(x_n)$ both exist, we have $y = \phi(x)$. Take $q(t) := |t| + |\phi(t)|$ for $t \in \mathbb{R}$.

   (i) Let (x_n) be a sequence, show that if there is a number L so that $\lim q(x_n - L) = 0$, then (x_n) is q-Cauchy.

   (ii) Is the number L in part (i) unique if it exists?

   (iii) Does the converse to part (i) hold true?

   Solutions.

   (i) Suppose (x_n) is a sequence and $L \in \mathbb{R}$ so that $\lim q(x_n - L) = 0$, then given arbitrary $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that for $n \geq N$, we have

   $$|q(x_n - L)| = |x_n - L| + |\phi(x_n - L)| < \frac{\epsilon}{2}.$$

   Therefore for the same N as above, and any $n, m \geq N$, we have

   $$\begin{aligned} |q(x_n - x_m)| &= |x_n - x_m| + |\phi(x_n - x_m)| \\ &= |(x_n - L) - (x_m - L)| + |\phi((x_n - L) - (x_m - L))| \\ &\leq |x_n - L| + |x_m - L| + |\phi(x_n - L)| + |\phi(x_m - L)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

   where in the above we have used additivity of ϕ and the triangle inequality. This shows that (x_n) is q-Cauchy.

   (ii) We claim that L is in fact equals to $\lim x_n$, hence is unique by uniqueness of limit of sequence. This follows directly from the observation that

   $$|x_n - L| \leq |x_n - L| + |\phi(x_n - L)| = |q(x_n - L)|.$$

   So if $\lim q(x_n - L) = 0$, we have $\lim x_n = L$, so such L must be unique.

   (iii) The converse holds true. Suppose (x_n) is a q-Cauchy sequence, again by a similar observation as above:

   $$|x_n - x_m| \leq |x_n - x_m| + |\phi(x_n - x_m)| = |q(x_n - x_m)|.$$

We know that (x_n) itself is a Cauchy sequence, therefore by Cauchy criterion (x_n) is convergent, with limit say $L := \lim x_n$. Also note that by the same argument, since $|\phi(x_n - x_m)| = |\phi(x_n) - \phi(x_m)|$, we know that $(\phi(x_n))$ is also a Cauchy sequence, and is convergent with limit $M := \lim \phi(x_n)$. By the assumption on the properties of ϕ, we have $\phi(L) = M$.

Now we claim that L given above satisfies $\lim q(x_n - L) = 0$. This is simply because $q(x_n - L) = |x_n - L| + |\phi(x_n - L)| = |x_n - L| + |\phi(x_n) - M|$, so by the convergence $\lim x_n = L$ and $\lim \phi(x_n) = M$, we have $\lim q(x_n - L) = 0$. ∎

Remark: As some of you pointed out, the function $\phi(x)$ satisfying the assumed properties must be of the form $\phi(x) = \alpha x$ for some $\alpha \in \mathbb{R}$, however this knowledge is not necessary to solve Q1. Exercise: Prove this. (Hint: Prove this for $x \in \mathbb{Q}$ first.)

2. (30 points)

   (i) Let (F_k) be a sequence of non-empty compact subsets of $\mathbb{R}$ so that $\bigcap_{k=1}^{n} F_k \neq \emptyset$ for any $n \in \mathbb{N}$, is it true that $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$?

   (ii) Let (J_k) be a sequence of closed and bounded intervals, suppose that $J_i \cap J_k \neq \emptyset$ for any $i, k \in \mathbb{N}$, is it true that $\bigcap_{k=1}^{\infty} J_k \neq \emptyset$?

   (iii) Is it possible to generalize the result of part (ii) to the two-dimensional case? That is, if $(A_k) = [a_k, b_k] \times [c_k, d_k]$ is a sequence of closed and bounded rectangles in $\mathbb{R}^2$, so that $A_i \cap A_k \neq \emptyset$ for any $i, k \in \mathbb{N}$, does it follow that $\bigcap_{k=1}^{\infty} A_k \neq \emptyset$?

Solutions.

(i) It is true. Denote $G_n := \bigcap_{k=1}^{n} F_k$. G_n is closed because it is an intersection of closed subsets: if $\{x_n\}$ is a convergent sequence in G_n, by closedness of F_k, $\lim x_n \in F_k$ for $k = 1, ..., n$, hence $\lim x_n \in G_n$ as well. G_n is also bounded because it is a subset of bounded F_1. Hence G_n is a non-empty compact subset. Now denote $s_n = \sup G_n$, since G_n is a decreasing sequence of subsets, (s_n) is also decreasing. It is bounded below by the lower bound of F_1. By monotone convergence theorem, $s := \lim s_n$ exists.

We claim that $s \in G_n$ for all $n \in \mathbb{N}$, therefore $s \in \bigcap_{n=1}^{\infty} G_n = \bigcap_{k=1}^{\infty} F_k$, thus proving that the intersection is non-empty. Fix any $n \in \mathbb{N}$, notice that for $m \geq n$, since G_m is closed, we know that $s_m = \sup G_m \in G_m \subset G_n$. The tail of the sequence (s_m) lies completely inside of G_n for any fixed n. In other words, the subsequence s'_k defined by $s'_k = s_{n+k}$ for $k \in \mathbb{N}$ is a subsequence contained entirely inside G_n, so $s = \lim s_m = \lim s'_k \in G_n$. This holds for arbitrary n.

(ii) The claim is true. By part (i), it suffices to prove that $\bigcap_{k=1}^{n} J_k$ is non-empty. Write $J_k = [a_k, b_k]$, notice that

$$\begin{aligned} x \in \bigcap_{k=1}^{n} J_k &\iff x \in J_k,\ \forall k \in \{1, ..., n\} \\ &\iff a_k \leq x \leq b_k,\ \forall k \in \{1, ..., n\} \\ &\iff \max_{1 \leq k \leq n} \{a_k\} \leq x \leq \min_{1 \leq k \leq n} \{b_k\}. \end{aligned}$$

Therefore, we observe that such x exists if and only if $\max_{1\leq k\leq n}\{a_k\} \leq \min_{1\leq k\leq n}\{b_k\}$. Suppose on the contrary that it was false, i.e. $\max_{1\leq k\leq n}\{a_k\} > \min_{1\leq k\leq n}\{b_k\}$, i.e. there are some distinct $1 \leq k, l \leq n$ so that $a_k > b_l$. Then it follows that $b_k \geq a_k > b_l \geq a_l$, and hence $J_k \cap J_l = \emptyset$, this is a contradiction. This proves that $\bigcap_{k=1}^{n} J_k$ is non-empty.

(iii) Write each $A_k = I_k \times J_k$ where I_k and J_k are closed and bounded intervals (in the first and second coordinates respectively), notice that $A_i \cap A_k \neq \emptyset$ if and only if $I_i \cap I_k \neq \emptyset$ and $J_i \cap J_k \neq \emptyset$. Therefore, by the results of part (ii), the intersections $I_\infty := \bigcap_{k=1}^{\infty} I_k$ and $J_\infty := \bigcap_{k=1}^{\infty} J_k$ are both non-empty. Pick any $x \in I_\infty$ and $y \in J_\infty$, then $(x, y) \in I_\infty \times J_\infty = \bigcap_{k=1}^{\infty} A_k$, so it is non-empty.