MATH 2058: Honours Mathematical Analysis I: Home Test 1
5:00 pm, 21 Oct 2022

Important Notice:

♣ The answer paper **must be submitted before 22 Oct 2022 at 5:00 pm.**
♠ The answer paper MUST BE sent to the CU Blackboard. After submitting the answer sheet, you ARE NOT Allowed to resubmit it again.
✠ **The answer paper must include your name and student ID.**

Answer ALL Questions

1. **(20 points)**
   For a non-negative function q on $\mathbb{R}$, we call a real sequence (x_n) a q-Cauchy sequence if for any $\varepsilon > 0$, there is a positive integer N such that $q(x_n - x_m) < \varepsilon$ for all $m, n \geq N$. Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a closed additive function, i.e., it satisfies the conditions: (i): $\phi(s+t) = \phi(s) + \phi(t)$ for all $s, t \in \mathbb{R}$; and (ii): whenever (x_n) is a sequence such that $x := \lim x_n$ and $y := \lim \phi(x_n)$ both exist, we have $\phi(x) = y$. Put

   $$q(t) := |t| + |\phi(t)| \qquad \text{for } t \in \mathbb{R}.$$

   (i) Show that if there is a number L such that $\lim_{n\to\infty} q(x_n - L) = 0$, then (x_n) is a q-Cauchy sequence.

   (ii) Is such L in Part (i) unique if it exists?

   (iii) Does the converse of Part (i) hold?

2. **(30 points)**

    (i) Let (F_k) be a sequence of non-empty compact subsets of $\mathbb{R}$. If $\bigcap_{k=1}^{n} F_k \neq \emptyset$ for all $n = 1, 2...$, does it imply that $\bigcap_{k=1}^{\infty} F_k$ is non-empty?

    (ii) Let (J_k) be a sequence of closed and bounded intervals. Assume that $J_i \cap J_k \neq \emptyset$ for all $i, k = 1, 2....$ Does it imply that $\bigcap_{k=1}^{\infty} J_k \neq \emptyset$?

    (iii) Is it possible to generalize the assertion in Part (ii) for the two dimensional case? More precisely, if (A_k) is a sequence of closed and bounded rectangles in $\mathbb{R}^2$, that is $A_k = [a_k, b_k] \times [c_k, d_k]$, and assume that $A_i \cap A_k \neq \emptyset$ for all $i, k = 1, 2...$, does it imply that $\bigcap_{k=1}^{\infty} A_k \neq \emptyset$?

$* * *$ **END OF PAPER** $* * *$