香 港 中 文 大 學
The Chinese University of Hong Kong

二零二二至二三年度上學期科目考試
Course Examination 1st Term, 2022-23

科目編號及名稱 Course Code & Title : **MATH2058 Honours Mathematical Analysis I**

時間 Time allowed : **2** 小時 hours **00** 分鐘 minutes

學號 Student I.D. No. : ______________ 座號 Seat No. : ______________

Answer ALL Questions

1. Let $h : (0, \infty) \longrightarrow \mathbb{R}$ be a function satisfying $\lim\limits_{t\to 0+} \dfrac{h(t)}{t} = 0$. Define a function $\varphi : (0,1) \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} ph(\frac{1}{q}) & \text{if } \ x = \frac{p}{q}, \ p, q \text{ are relatively prime positive integers;} \\ 0 & \text{otherwise.} \end{cases}$$

   (i) Show that for every $\varepsilon > 0$, the set $N(\varepsilon) := \{x \in (0,1) : |\varphi(x)| \geq \varepsilon\}$ is finite.

   (ii) Show that the function φ is continuous at every irrational point in $(0,1)$.

2. For each $x = (x_1, ..., x_m) \in \mathbb{R}^m$, put $\|x\| := \sqrt[2]{x_1^2 + \cdots + x_m^2}$. Now $\mathbb{R}^m$ is endowed with the usual metric, i.e., the distance between the elements x and y in $\mathbb{R}^m$ is given by $\|x - y\|$.
Let $q(x) := \sqrt[3]{|x_1|^3 + \cdots + |x_m|^3}$ and let A be the set $\{x \in \mathbb{R}^m : q(x) = 1\}$.

   (i) Show that the set A is compact.

   (ii) Show that there are $c_1, c_2 > 0$ such that $c_1 q(x) \leq \|x\| \leq c_2 q(x)$ for all $x \in \mathbb{R}^m$.

3. Prove or disprove the following statements.

   (i) There is a continuous function f defined on the set $A := \bigcup\limits_{n=1}^{\infty} \left[\dfrac{1}{2n+1}, \dfrac{1}{2n}\right] \cup \{0\}$ so that the image of f is the set $\{\dfrac{1}{n} : n = 1, 2, ...\}$.

   (ii) For every positive integer n, there is a continuous real valued function g defined on $D := (0,1) \cap \mathbb{Q}$ so that the image of g is $\{1, 2, ..., n\}$

4. Let A be a non-empty subset of $\mathbb{R}$. For a function $f : A \longrightarrow \mathbb{R}$, put $\omega_f(t) := \sup\{|f(u) - f(v)| : u, v \in A; \ |u - v| < t\}$ provided the supremum exists for some $t > 0$.

   (i) Show that if there is $c > 0$ such that $\omega_f(t) \leq ct$, for all $t > 0$, then f is uniformly continuous on A.

   (ii) Let f be a function defined on $[0,1]$ given by $f(x) := x \sin \frac{1}{x}$ for $x \in (0,1]$ and $f(0) = 0$. Use the function f to show that the converse of Part (i) does not hold.

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