MATH 2050C Lecture 22 (Apr 11)

"Uniform" Continuity (§ 5.4 in textbook)

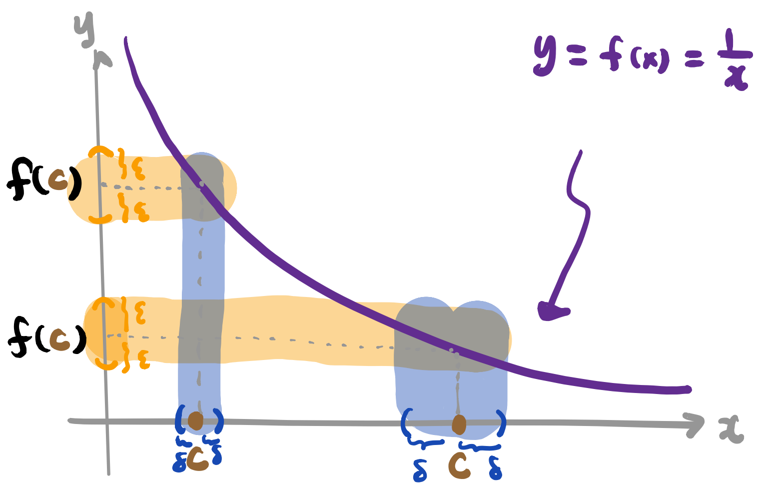
Recall: Let $f: A \to \mathbb{R}$.

- f is cts at $c \in A$ $\iff$ $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.
$$|f(x) - f(c)| < \varepsilon \quad \forall |x - c| < \delta$$

- f is cts on A $\overset{\text{def}^{\text{n}}}{\iff}$ f is cts at EVERY $c \in A$
$$\iff \forall c \in A, \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon, c) > 0,$$
$$\text{s.t. } |f(x) - f(c)| < \varepsilon \quad \forall |x - c| < \delta$$

Caution: The choice of δ depends on BOTH ε AND c.

Example: $f: (0, \infty) \to \mathbb{R}$ $\quad f(x) := \frac{1}{x}$ cts on $(0, \infty)$



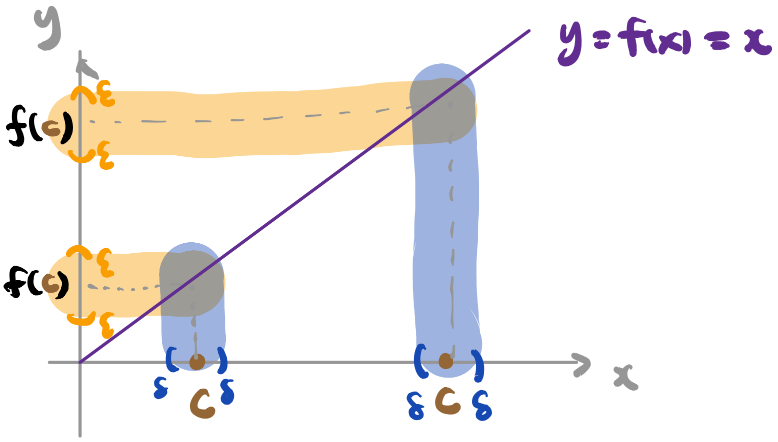
FOR THE SAME $\varepsilon > 0$

If $c \approx 0$, then we need to choose a much smaller δ s.t.
$$|f(x) - f(c)| < \varepsilon \quad \forall |x - c| < \delta$$

Idea: This function is NOT "uniformly" cts

$\because$ δ is NOT "uniform" in c

Example: $f : (0,\infty) \to \mathbb{R}$ $f(x) := x$ cts on $(0,\infty)$



FOR THE SAME $\varepsilon > 0$

You can choose ONE $\delta > 0$

s.t. it works for ALL $c \in A$

$|f(x) - f(c)| < \varepsilon \quad \forall\ |x - c| < \delta$

Idea: This function is "uniformly" cts.

Def^n: $f : A \to \mathbb{R}$ is uniformly continuous (on A)

iff $\forall\, \varepsilon > 0,\ \exists\, \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(u) - f(v)| < \varepsilon \quad \forall\ u, v \in A,\ |u - v| < \delta$$

Remark: (1) uniform cts $\Rightarrow$ cts on A ($\because$ take $v = c \in A$)

(2) Uniform continuity is a "global" concept. It does NOT make sense to talk about uniform continuity at one point $c \in A$.

Q: How to see if $f : A \to \mathbb{R}$ is uniformly cts (on A)?

We first begin with a "non-uniform continuity" criteria.

Prop: $f: A \to \mathbb{R}$ is NOT uniformly continuous

$\iff \exists\, \varepsilon_0 > 0$ st $\forall\, \delta > 0$, $\exists\, u_\delta, v_\delta \in A$

st $|u_\delta - v_\delta| < \delta$ BUT $|f(u_\delta) - f(v_\delta)| \geq \varepsilon_0$

$\iff \exists\, \varepsilon_0 > 0$ and seq. $(u_n), (v_n)$ in A

st $|u_n - v_n| < \frac{1}{n}$ BUT $|f(u_n) - f(v_n)| \geq \varepsilon_0$ $\forall n \in \mathbb{N}$

Proof: Take negation of def^{n} and choose $\delta = \frac{1}{n}$. $\blacksquare$

Example: Show that $f: (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, is NOT uniformly continuous on $(0,\infty)$.

Proof: Take $(u_n) := (\frac{1}{n})$ and $(v_n) := (\frac{1}{n+1})$ in $(0,\infty)$.

THEN, $|u_n - v_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n}$ $\forall n \in \mathbb{N}$

BUT $|f(u_n) - f(v_n)| = |n - (n+1)| = 1 \geq \varepsilon_0 := \frac{1}{2} > 0$.

By Prop, f is NOT uniformly cts on $(0,\infty)$. $\blacksquare$

Exercise: Show that $f: [a,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is uniformly cts on $[a,\infty)$ for any fixed $a > 0$.

Idea: We can say more about "uniform continuity" of $f: A \to \mathbb{R}$ if A is an interval.

- Uniform Continuity Thm
- Continuous Extension Thm

Uniform Continuity Thm

closed & bdd interval.

$f: [a,b] \to \mathbb{R}$ cts on $[a,b]$ $\Rightarrow$ f is uniformly cts on $[a,b]$.

Proof: Argue by contradiction. Suppose NOT, i.e. f is NOT uniformly cts. Then, by non-uniform continuity criteria, $\exists\ \varepsilon_0 > 0$ and seq. (u_n), (v_n) in $[a,b]$

$(*)$ .... [s.t. $|u_n - v_n| < \frac{1}{n}$ BUT $|f(u_n) - f(v_n)| \geq \varepsilon_0 \quad \forall n \in \mathbb{N}$]

By Bolzano-Weierstrass Thm, since (u_n) is bdd.

$\Rightarrow \exists$ subseq. (u_{n_k}) of (u_n) s.t.

$$\lim_{k\to\infty} (u_{n_k}) = x^* \in [a,b]$$

Claim: $\lim_{k\to\infty} (v_{n_k}) = x^*$

Pf: $|u_{n_k} - v_{n_k}| < \frac{1}{n_k}$ $\forall k \in \mathbb{N}$ $\xRightarrow{\text{as } k\to\infty}$ $\lim_{k\to\infty} (v_{n_k}) = x^*$ by limit thm.

By continuity of f at $x^* \in [a,b]$.

$$0 < \varepsilon_0 \overset{(*)}{\leq} \lim_{k\to\infty} |f(u_{n_k}) - f(v_{n_k})| = |f(x^*) - f(x^*)| = 0$$

Contradiction! $\square$

Q: When can we extend a cts $f: (a,b) \to \mathbb{R}$ to a cts function $\bar{f}: [a,b] \to \mathbb{R}$?

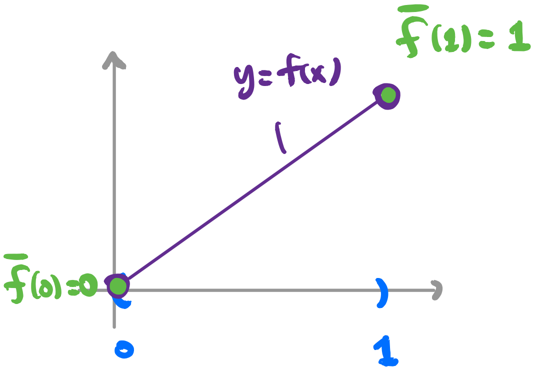
(s.t. $\bar{f}(x) = f(x) \quad \forall x \in (a,b)$.)

Examples:

Yes!

$f(x) = x$ on $x \in (0,1)$

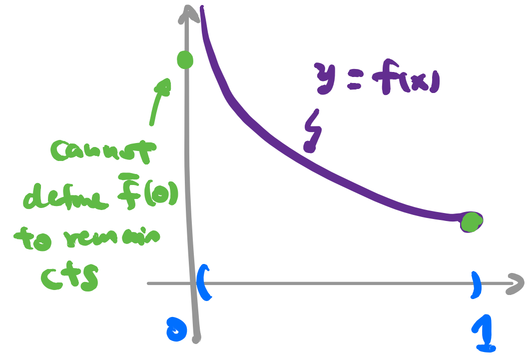
extend ⇝ $\bar{f}(x) = x$ on $x \in [0,1]$



No!

$f(x) = \frac{1}{x}$ on $x \in (0,1)$

⇝ $\nexists$ cts extension $\bar{f}$ to $[0,1]$.



Continuous Extension Thm

If $f : (a,b) \to \mathbb{R}$ is uniformly cts (on (a,b))

then $\exists$ an "extension" $\bar{f} : [a,b] \to \mathbb{R}$ s.t.

(i) $\bar{f}(x) = f(x) \quad \forall x \in [a,b]$

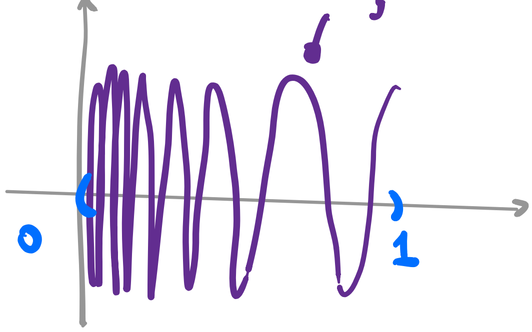
(ii) $\bar{f}$ is cts on $[a,b]$

Remarks: (a) $\bar{f}$ is uniformly cts by Uniform Continuity Thm

(b) Such an extension $\bar{f}$ is unique.

Example:

$y = f(x) = \sin\frac{1}{x}$ remain bdd on $(0,1)$



BUT not unif. cts on $(0,1)$

[Ex: Prove this.]

We will use the following lemma in the proof.

<u>Lemma</u>: Let $f: A \to \mathbb{R}$ be uniformly cts. THEN,

$$(x_n) \text{ Cauchy seq in } A \implies (f(x_n)) \text{ Cauchy seq. in } \mathbb{R}$$

<u>Proof of Lemma</u>: Let $\varepsilon > 0$.

By uniform continuity of f, $\exists\ \delta = \delta(\varepsilon) > 0$ s.t.

$(\#)\cdots \left[\ |f(u) - f(v)| < \varepsilon \quad \text{whenever } u, v \in A \text{ st } |u - v| < \delta\ \right]$

Let (x_n) be a Cauchy seq in A. By ε-H def^n,

for this $\delta > 0$ above, $\exists\ H = H(\delta) \in \mathbb{N}$ s.t.

$$|x_m - x_n| < \delta \quad \forall\, n, m \geq H$$

By $(\#)$, $|f(x_m) - f(x_n)| < \varepsilon \quad \forall\, n, m \geq H$

So, $(f(x_n))$ is Cauchy.

$\square$