MATH 2050C Lecture 21 (Apr 6)

Three important theorems about continuous $f : [a, b] \to \mathbb{R}$

- Boundedness Thm
- Extreme Value Theorem
- Intermediate Value Theorem

[compactness]

Extreme Value Thm: A cts $f : [a, b] \to \mathbb{R}$ always achieve its absolute maximum and minimum, i.e.

$$\exists\, x^* \in [a, b] \text{ s.t. } f(x^*) = M := \sup\{f(x) \mid x \in [a, b]\}$$

$$\exists\, x_* \in [a, b] \text{ s.t. } f(x_*) = m := \inf\{f(x) \mid x \in [a, b]\}$$

not nec. unique

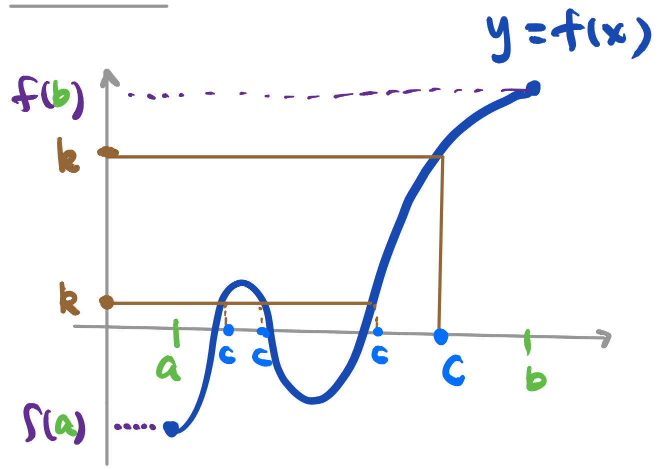
Intermediate Value Theorem [connectedness]

Let $f : [a, b] \to \mathbb{R}$ be a cts function s.t. $f(a) < f(b)$.

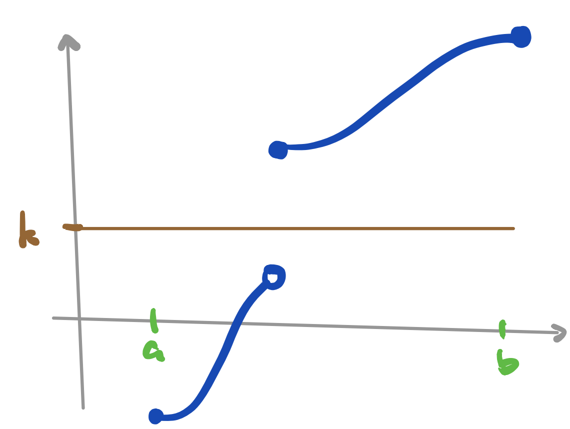
THEN, $\forall\, k \in (f(a), f(b))$, $\exists\, c \in [a, b]$ s.t.

$$f(c) = k$$

Picture:

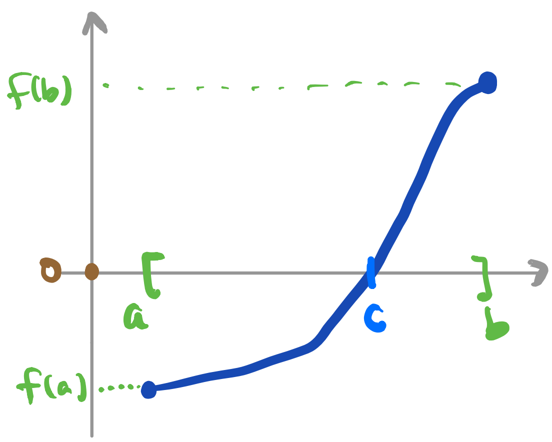


Continuity is needed



Proof: It suffices to consider the case:

$$f(a) < 0 < f(b) \quad \text{and} \quad k = 0$$



[$\because$ The general case follows by considering $g(x) := f(x) - k$.]

$g(c) = 0 \iff f(c) = k$

Q: How to locate a "root" where $f(c) = 0$?

A: Method of bisection.

Define a nested seq of closed & bdd intervals I_n as follows.

Take $I_1 := [a, b] =: [a_1, b_1]$

Consider the midpt. $\frac{a_1+b_1}{2}$ of I_1

Case 1: $f\left(\frac{a_1+b_1}{2}\right) < 0 \Rightarrow$ take $I_2 := [a_2, b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$

Case 2: $f\left(\frac{a_1+b_1}{2}\right) > 0 \Rightarrow$ take $I_2 := [a_2, b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$

Case 3: $f\left(\frac{a_1+b_1}{2}\right) = 0 \Rightarrow$ DONE, take $c = \frac{a_1+b_1}{2}$.

Repeat this process for I_2.

Either you locate a root (Case 3), or you obtain a seq. of closed & bdd intervals $I_n := [a_n, b_n]$. $\text{Length}(I_{n+1}) = \frac{1}{2}\text{Length}(I_n)$

st $\begin{cases} I_{n+1} \subseteq I_n & \forall n \in \mathbb{N} \quad \underline{\text{nested}} \\ f(a_n) < 0 < f(b_n) & \forall n \in \mathbb{N}. \quad \text{———} \ (\#) \end{cases}$

By Nested Interval Property, $\bigcap_{n=1}^{\infty} I_n = \{c\}$

($\because \text{Length}(I_n) \to 0$)

Claim: $f(c) = 0$.

Pf: Since $\lim(a_n) = \lim(b_n) = c$, take $n \to \infty$ in (*), by continuity of f at c,

$$f(c) \leq 0 \leq f(c), \text{ ie } f(c) = 0$$

$\square$