MATH 2050C Lecture 20 (Apr 4)

Last Quiz on Apr 6, covers § 5.1 – 5.2.

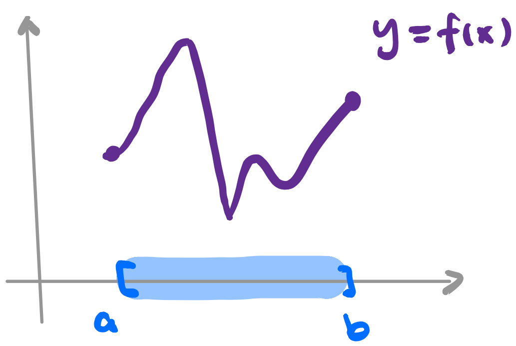
[Problem Set 11 posted, due on Apr 14]

Last time ..... continuity of $f : A \to \mathbb{R}$ at $c \in A$ (or $B \subseteq A$)

Q: What about if A = interval, can we say more?

§ Continuous functions on intervals (§ 5.3 in textbook)

Q: What can we say about cts fcn $f : [a,b] \to \mathbb{R}$? ($[a,b]$ = A closed & bdd interval)



Note: All points $c \in [a,b]$ are cluster points of $[a,b]$.

i.e. $\lim_{x \to c} f(x) = f(c)$

In terms of ε-δ def^{n}.

$\forall c \in [a,b]$, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, c) > 0$

s.t. $|f(x) - f(c)| < \varepsilon$ when $|x - c| < \delta$, $x \in [a,b]$

Recall: f cts at $c \Rightarrow f$ is "locally bdd" near c

Boundedness Thm: Any cts $f : [a,b] \to \mathbb{R}$ is bdd (globally on $[a,b]$)

i.e. $\exists M > 0$ st $|f(x)| \leq M \quad \forall x \in [a,b]$.

Proof: Argue by contradiction. Suppose f is NOT bdd on $[a,b]$.

$\Rightarrow \forall n \in \mathbb{N}$, $\exists x_n \in [a,b]$ st $|f(x_n)| > n$ .... $(*)$

We obtain a seq. (x_n) in $[a,b]$, hence is bdd.

By Bolzano-Weierstrass Thm, $\exists$ convergent subseq. (x_{n_k}) of (x_n)

say $\lim_{k\to\infty}(x_{n_k}) =: x_*$

Now, $a \le x_{n_k} \le b \quad \forall k \in \mathbb{N} \overset{\text{limit}}{\underset{\text{thm}}{\Longrightarrow}} a \le \lim (x_{n_k}) \le b$

ie $x_* \in [a,b]$.

$\lim_{x\to x_*} f(x) = f(x_*)$

By continuity of f at x_*, and seq. criteria, "$\lim_{x\to x_*} f(x) = \lim_{k\to\infty}(f(x_{n_k}))$"

$$\lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(x_*)$$

So, $f(x_{n_k}) \to f(x_*)$ as $k\to\infty \Rightarrow (f(x_{n_k}))$ is bdd.

contradiction

However, $\underbrace{|f(x_{n_k})| > n_k \ge k}_{\text{by construction } (*)} \quad \forall k\in\mathbb{N} \Rightarrow (f(x_{n_k}))$ is unbdd.

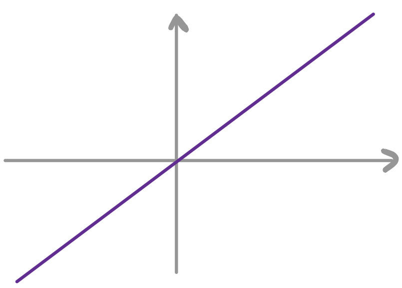
$\square$

Remark: All assumptions are required in the theorem.

(1) unbdd interval

$f: \mathbb{R} \to \mathbb{R}$

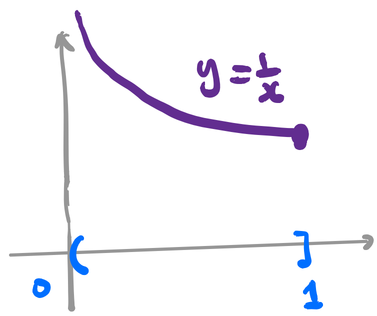
$f(x) := x$



(2) non-closed interval

$f: (0,1] \to \mathbb{R}$

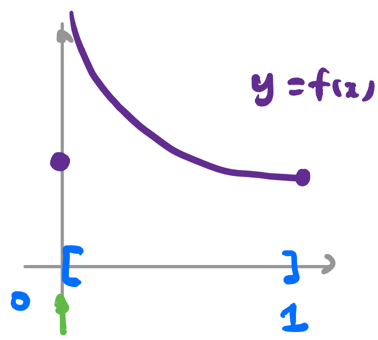
$f(x) := \frac{1}{x}$



(3) not continuity

$f: [0,1] \to \mathbb{R}$

$f(x) := \begin{cases} 1/x, & \text{if } x \ne 0 \\ 1, & \text{if } x = 0. \end{cases}$



f is NOT cts at 0

By Boundedness Theorem, $\exists$ exist in $\mathbb{R}$

$$M := \sup\{f(x) \mid x \in [a,b]\}$$

$$m := \inf\{f(x) \mid x \in [a,b]\}$$

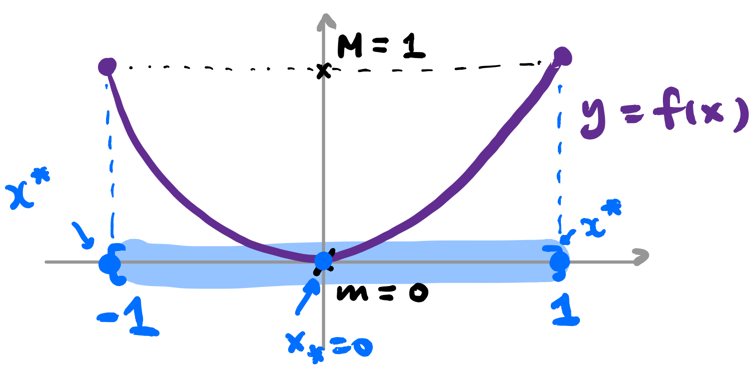
Extreme Value Thm: A cts $f : [a,b] \to \mathbb{R}$ always achieve its maximum and minimum, i.e.

$\exists\ x^* \in [a,b]$ st $f(x^*) = M := \sup\{f(x) \mid x \in [a,b]\}$

not nec. unique

$\exists\ x_* \in [a,b]$ st $f(x_*) = m := \inf\{f(x) \mid x \in [a,b]\}$

Example: $f(x) = x^2$, $f : [-1, 1] \to \mathbb{R}$



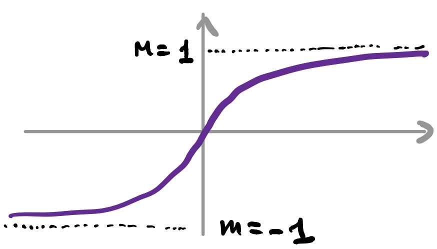
Caution: There can be more than one maxima x^* and minima x_*.

Remarks: All assumptions are required.

(1) unbdd interval

$f : \mathbb{R} \to \mathbb{R}$

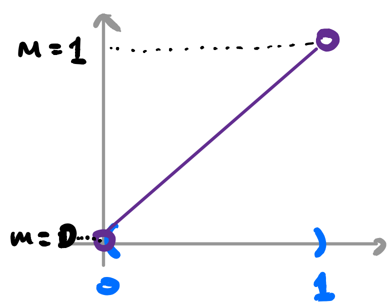
$f(x) = \tanh x$



(2) non-closed interval

$f : (0,1) \to \mathbb{R}$

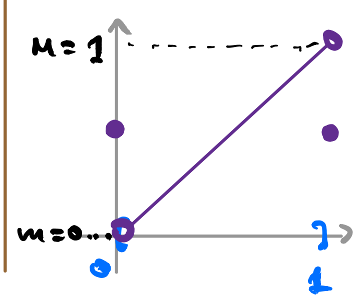
$f(x) = x$



(3) not cts

$f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 1/2 & \text{if } x = 0, 1 \end{cases}$$



Proof: We only prove the existence of x^*.

Since $M := \sup\{f(x) \mid x \in [a,b]\}$, $\forall \varepsilon > 0$, $\exists\, x_\varepsilon \in [a,b]$ st.

$$M - \varepsilon < f(x_\varepsilon)$$

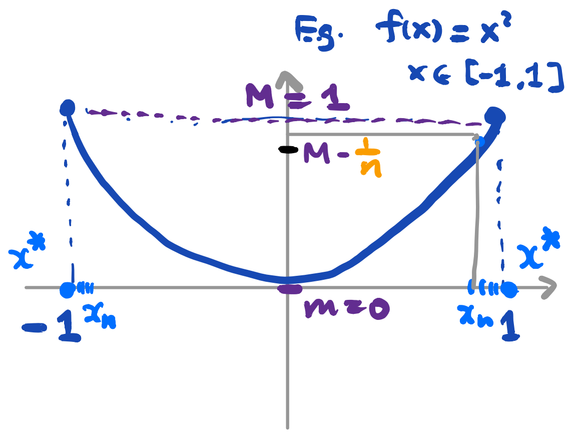
Take $\varepsilon = \frac{1}{n}$, then we obtain a sequence $(x_n) \subseteq [a,b]$ st.

$$M - \frac{1}{n} < f(x_n) \leq M$$

By Bolzano-Weierstrass Thm, since (x_n) is a bdd seq.

$\Rightarrow \exists$ convergent subseq. (x_{n_k}) of (x_n), say $\underset{[a,b]}{\underset{\Uparrow}{x^*}} := \lim(x_{n_k})$

Claim: $f(x^*) = M$



Pf: Since $M - \frac{1}{n_k} < f(x_{n_k}) \leq M$

for all $k \in \mathbb{N}$,

take $k \to \infty$, by continuity of f at x^*

$$M \leq f(x^*) = \lim_{k\to\infty} f(x_{n_k}) \leq M$$

Limit theorems

$\square$