MATH 2050C Lecture 18 (Mar 28)

Last time ...... Sequential Criteria, Divergence criteria, & limit theorems

ASSUME: Functions are defined on $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ is a cluster pt of A.

Squeeze / Sandwich Thm (for functions)

Let $g, f, h : A \to \mathbb{R}$ be functions s.t.

$$g(x) \leq f(x) \leq h(x) \quad \forall x \in A \quad \cdots\cdots (\dagger)$$

Suppose $\lim\limits_{x \to c} g(x) = L = \lim\limits_{x \to c} h(x)$.

THEN. $\lim\limits_{x \to c} f(x) = L$.

Remarks: 1) The existence of $\lim\limits_{x \to c} f(x)$ is a conclusion

2) One only requires $(\dagger)$ to hold in some neighborhood of c.

Proof: Use sequential criteria.

Let (x_n) be a sequence in A s.t. $x_n \neq c$ $\forall n \in \mathbb{N}$. $\lim (x_n) = c$.

Claim: $\lim f(x_n) = L$

Pf: By $(\dagger)$, $g(x_n) \leq f(x_n) \leq h(x_n)$ $\forall n \in \mathbb{N}$

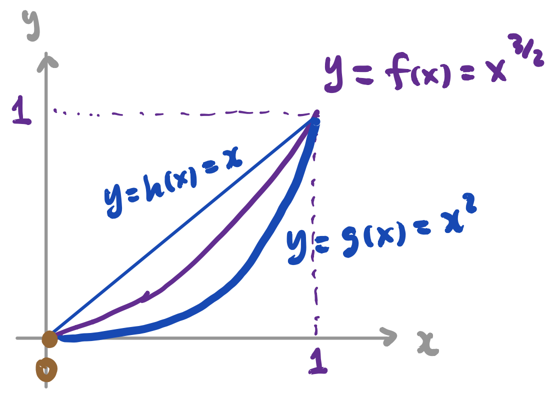
By Seq. Criteria. $\lim g(x_n) = L = \lim h(x_n)$.

By Squeeze Thm for seq., $\lim f(x_n) = L$.

$\square$

Example 1 : $\boxed{\lim_{x\to 0} x^{3/2} = 0}$

Proof: Here: $f : A := \{x \in \mathbb{R} \mid x \geq 0\} \to \mathbb{R}$ where $f(x) := x^{3/2}$.



Take $g, h : A \to \mathbb{R}$ as

$$g(x) = x^2 \quad \& \quad h(x) = x.$$

Note that

$$x^2 \leq x^{3/2} \leq x \qquad \forall x \in [0, 1]$$

By squeeze thm. since

$$\lim_{x\to 0} x^2 = 0 = \lim_{x\to 0} x$$

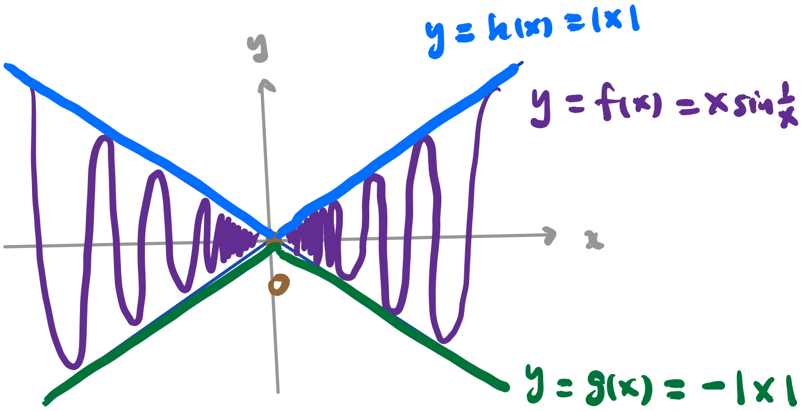
So $\lim_{x\to 0} x^{3/2} = 0$.

$\square$

Example 2 : $\boxed{\lim_{x\to 0} x \sin\frac{1}{x} = 0}$

(Recall: $\lim_{x\to 0}\left(\sin\frac{1}{x}\right)$ DOES NOT EXIST by seq. criteria.)

Proof: Here: $f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$, and $f(x) = x \sin\frac{1}{x}$.



Since $|\sin\frac{1}{x}| \leq 1$, we have

$$-|x| \leq x \sin\frac{1}{x} \leq |x| \quad \forall x \in A$$

Now $\lim_{x\to 0} |x| = 0 = \lim_{x\to 0} -|x|$

By Squeeze Thm.

$$\lim_{x\to 0} x \sin\frac{1}{x} = 0.$$

$\square$

Prop: Suppose $\lim_{x\to c} f(x) = L > 0$. THEN, $\exists\, \delta > 0$ s.t.

$$f(x) > 0 \qquad \forall\, x \in A \text{ s.t. } 0<|x-c|<\delta$$

Remark: The Prop. DOES NOT hold if we replace $>$ by $\geq$.

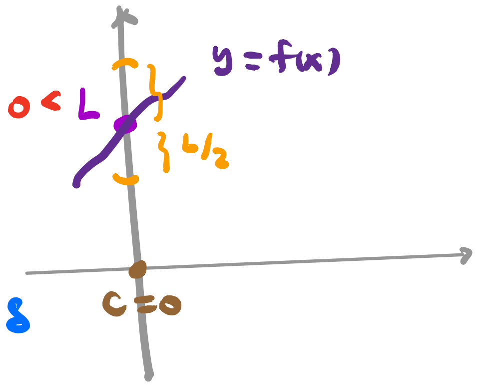
e.g. $L = 0$ (see Example 2 above)

Proof: Use ε-δ def^{n} !

Take $\varepsilon := L/_2 > 0$.

Then $\exists\, \delta = \delta(\frac{L}{2}) > 0$ s.t.

$$|f(x) - L| < \varepsilon = \frac{L}{2} \qquad \forall\, 0<|x-c|<\delta$$



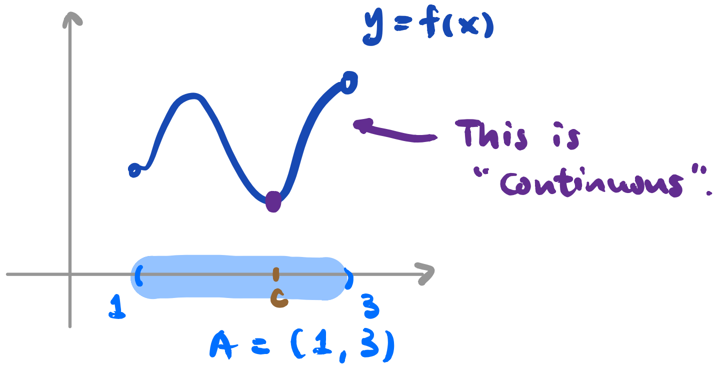
$$\Rightarrow \qquad f(x) \geq L - \frac{L}{2} = \frac{L}{2} > 0 \qquad \forall\, 0<|x-c|<\delta$$

□

§ Continuity of functions (Ch.5)

Q: What does "continuity" mean?

$f : A \to \mathbb{R}$



A: "f is continuous at c"

$\iff$ "$f(x) \underset{\varepsilon}{\approx} f(c)$ when $x \underset{\delta}{\approx} c$"

Note: We NEED $c \in A$.

Def$^{\underline{n}}$: (ε-δ def$^{\underline{n}}$ for continuity)

Given $f: A \to \mathbb{R}$ and $c \in A$, we say that "f is (cts) continuous at c" if $\forall\, \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$ s.t.

(*) …… $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$, $|x - c| < \delta$

Remark: Compared to the def$^{\underline{n}}$ of $\lim\limits_{x \to c} f(x) = L$, we have

- L is replaced by $f(c) \Rightarrow c \in A$
- $f(c)$ matters here, unlike $\lim\limits_{x \to c} f(x) = L$
- (*) is always satisfied at $x = c$
- c may or may not be a cluster point of A

For the last remark,

Case 1: When c **IS** a cluster pt. of A

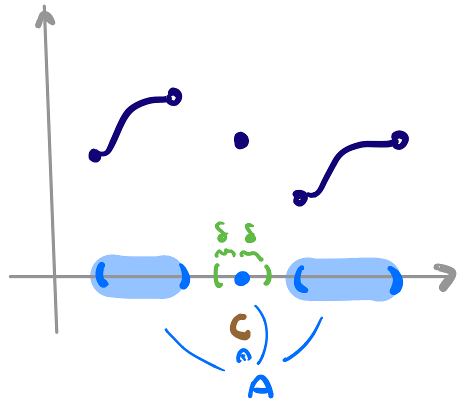
"f is cts at $c \in A$" $\Longleftrightarrow$ "$\lim\limits_{x \to c} f(x) = f(c)$"

interesting case

ie you can "substitute" to evaluate the limit at c.

Case 2: when c is **NOT** a cluster pt. of A

Then, f is always cts at $c \in A$



Why? In this case, $\exists\, \delta > 0$ s.t.

$$A \cap (c - \delta, c + \delta) = \{c\}$$

$\Rightarrow$ (*) is trivially satisfied.

Note: "continuity" is a pointwise condition.

Def^n: $f: A \to \mathbb{R}$ is continuous on a subset $B \subseteq A$

if f is continuous at EVERY $c \in B$.

In particular, if $B = A$, then we say f is continuous (everywhere).

Examples of continuous functions

- $f(x) = b$ constant function
- $f(x) = x$ or $f(x) = x^2$
- $f(x) = p(x)$ polynomial function
- $f(x) = \sin x$ or $\cos x$ or $\tan x$
- $f(x) = e^x$ or $\sqrt{x}$

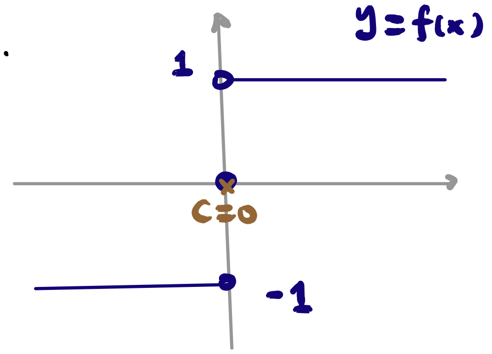
Example of dis-continuous functions

Example 1: Consider $f: \mathbb{R} = A \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

"sign function"

Show that f is NOT cts at $x = 0$.



Proof: Note $0 \in A$ is a cluster pt. of $A = \mathbb{R}$.

Check whether $\lim\limits_{x \to 0} f(x) \overset{?}{=} f(0)$

In this case $\lim\limits_{x \to 0} f(x)$ DOES NOT EXIST!

Consider $(x_n) = \left(\frac{(-1)^n}{n}\right) \to 0$ and

note $(f(x_n)) = ((-1)^n)$ is divergent $\Bigg\} \overset{\text{Seq}}{\underset{\text{Criteria}}{\Longrightarrow}} \lim\limits_{x \to 0} f(x)$ does not exist.

□

Remark: For this f, it is dis continuous at 0 no matter what the value of $f(0)$ is.

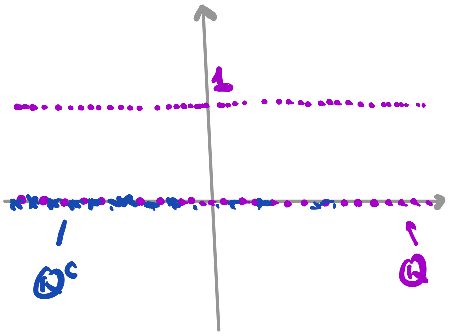
Example 2: The function $f: A = \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is dis-continuous EVERYWHERE.

(#)

Proof: Key idea: Density of $\mathbb{Q}$ or $\mathbb{Q}^c$ in $\mathbb{R}$.



Take $c \in \mathbb{R}$. There are 2 cases:

Case 1: $c \in \mathbb{Q}$.

Claim: $\lim\limits_{x \to c} f(x)$ DOES NOT EXIST.

Reason:
$\exists$ rational numbers $(\overset{\overset{c}{\#}}{x_n}) \to c \Rightarrow (f(x_n)) = (1) \to 1$

$\#$

$\exists$ irrational numbers $(\overset{\overset{c}{\#}}{x_n'}) \to c \Rightarrow (f(x_n')) = (0) \to 0$

density (#)

DONE by Seq. criteria!

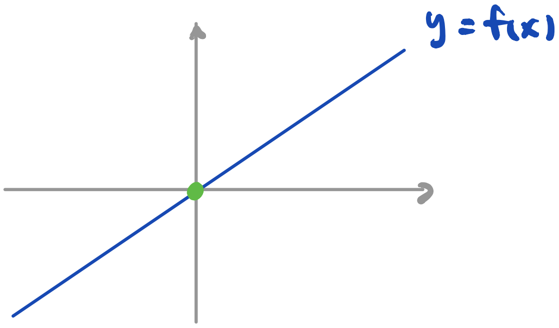
Case 2: $c \notin \mathbb{Q}$ is the same.

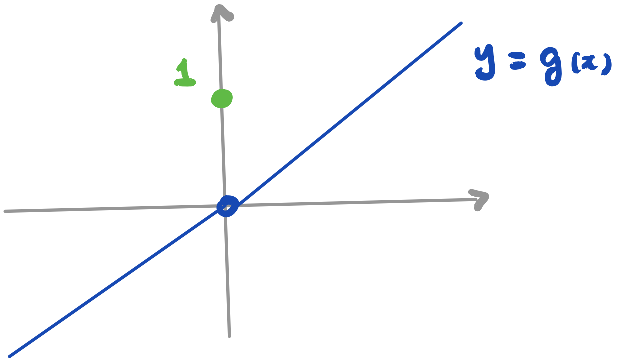
∎

Recall: Continuity of f at $c \in A$ is sensitive to the value of $f(c)$.

Example: [Sometimes you can make a fcn cts by redefining it at a pt.]

$$f(x) := \begin{cases} x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \underset{\text{only at } x=0}{\overset{\text{differs}}{\longleftrightarrow}} \quad g(x) := \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$





This is cts at 0. This is NOT cts at 0.

BUT: $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x)$

- More complicated examples in the tutorial / exercise.