MATH 2050 Lecture 17 (Mar 23, 2023)

[Problem Set 9 posted, due on Mar 31, 2023)

Limit Theorems for functions (§ 4.2 in textbook)

(iff statement)

Motto: By Sequential Criteria, we get limit theorems for functions from the corresponding limit theorems for sequences

Recall: For sequences, (x_n) convergent $\Rightarrow$ (x_n) bdd

We have a similar result for functions.

Boundedness Thm:

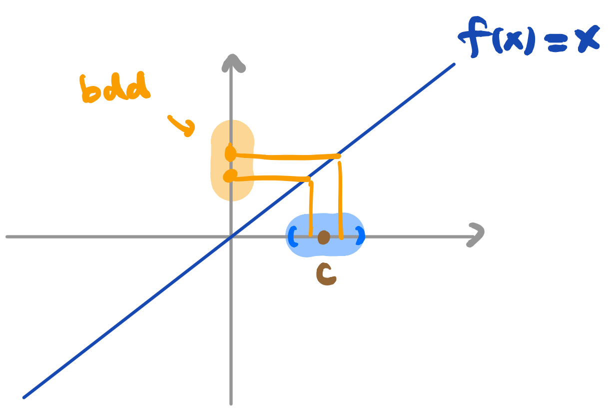
$\lim_{x \to c} f(x)$ exists $\Rightarrow$ f is "bdd in a neighborhood of c"

i.e. $\exists M > 0$ and $\exists \delta > 0$ s.t.

(Note: $\Leftarrow$ not true) $\quad |f(x)| \leq M \quad \forall \, |x - c| < \delta$ and $x \in A$

Remark: f may not be bdd "globally".

E.g.)



Proof: By ε-δ def^{n}, $\lim_{x \to c} f(x) = L$ $\Rightarrow$ Take $\varepsilon = 1$

Then $\exists \, \delta = \delta(1) > 0$ s.t. $|f(x) - L| < \varepsilon = 1$

whenever $x \in A$ and $0 < |x - c| < \delta$

by Δ-ineq
$$\Rightarrow \begin{cases} |f(x)| \leq |f(x) - L| + |L| < 1 + |L| \\ \text{whenever } x \in A \text{ and } 0 < |x-c| < \delta \end{cases}$$

If we take $M := \max\{1 + |L|, \underbrace{|f(c)|}_{\text{if } c \in A}\} > 0$, then

we have $|f(x)| \leq M \quad \forall x \in A$ s.t $|x - c| < \delta$

$\blacksquare$

Def$^{\underline{n}}$: Given $f, g : A \to \mathbb{R}$ functions defined on the same A, then we can define new functions:

- $(f \pm g)(x) := f(x) \pm g(x)$ $\qquad f \pm g : A \to \mathbb{R}$
- $(fg)(x) := f(x)\, g(x)$ $\qquad fg : A \to \mathbb{R}$
- $\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$ $\qquad \frac{f}{g} : A \setminus \{x \in A \mid g(x) = 0\} \to \mathbb{R}$

Thm: (1) $\lim_{x \to c} (f \pm g)(x) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

(2) $\lim_{x \to c} (fg)(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

**(3) $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$

extra careful!

provided that $\lim_{x \to c} f(x)$, $\lim_{x \to c} g(x)$ exist,

and for (3), additionally, we need $\lim_{c \to c} g(x) \neq 0$

Examples: $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$ provided $c \neq 0$; $\lim_{x\to 3} \frac{x^3-4}{x+1} = \frac{4}{3}$

$$\lim_{x\to 2} \frac{x^2-4}{3x-6} = \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{3\cancel{(x-2)}} = \frac{4}{3}.$$

Proof of (2): IDEA: Use seq criteria.

Take (x_n) in A s.t. $x_n \neq c$ $\forall n \in \mathbb{N}$ and $\lim(x_n) = c$.

Seq criteria $\Rightarrow$ $(f(x_n)) \to \lim_{x\to c} f(x)$ & $(g(x_n)) \to \lim_{x\to c} g(x)$

Limit Thm for seq. $\Rightarrow$ $(f(x_n)\cdot g(x_n)) \to \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

Seq criteria $\Rightarrow$ $\lim_{x\to c} (fg)(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

$\square$