MATH 2050C Lecture 16 (Mar 21)

[Quiz 3 on Mar 23, 2023.]

Recall: " ε-δ definition for limit of functions" $f : A \to \mathbb{R}$

$\lim\limits_{x \to c} f(x) = L \iff \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.

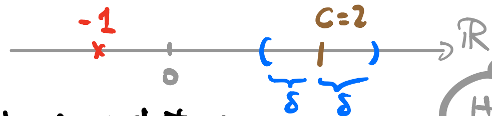
$|f(x) - L| < \varepsilon$ whenever $x \in A$ and $0 < |x - c| < \delta$

(c: cluster point)

Example: $\boxed{\lim\limits_{x \to 2} \dfrac{x^3 - 4}{x+1} = \dfrac{4}{3}}$

$f : A := \mathbb{R} \setminus \{-1\} \to \mathbb{R}$

$f(x) := \dfrac{x^3-4}{x+1}$



Pf: Let $\varepsilon > 0$ be fixed but arbitrary.

Note: If $|x - 2| < 1$, then

$1 < x < 3$

Hence, $|x+1| > 2 > 0$

and $|3x^2 + 6x + 8| \leq 1000$.

Choose $\delta = \min\{1, \frac{3}{500}\varepsilon\}$.

THEN, $\forall x \in A$, and $0 < |x - 2| < \delta$,

$$\left|\frac{x^3-4}{x+1} - \frac{4}{3}\right| = \left|\frac{3x^3 - 4x - 16}{3(x+1)}\right|$$

$$= \frac{|3x^2+6x+8|}{3|x+1|} \cdot |x-2|$$

$$< \frac{1000}{3 \cdot 2} \delta \leq \varepsilon$$

$\blacksquare$

Here: $0 < |x-2| < \delta$

$$\left|\frac{x^3-4}{x+1} - \frac{4}{3}\right| = \left|\frac{3(x^3-4) - 4(x+1)}{3(x+1)}\right|$$

$$= \left|\frac{3x^3 - 4x - 16}{3(x+1)}\right|$$

$$= \left|\frac{(x-2)(3x^2+6x+8)}{3(x+1)}\right|$$

$$= \frac{|3x^2+6x+8|}{3|x+1|} \cdot \underbrace{|x-2|}_{\text{small}}$$

Note: $|x-2| < 1$

$\Rightarrow 1 < |x| < 3$

So $|x+1| > 2 > 0$.

and $|3x^2 + 6x + 8|$

$\leq 3|x|^2 + 6|x| + 8$

$\leq 3 \cdot 3^2 + 6 \cdot 3 + 8 \leq 1000$

Prop: $\lim_{x\to c} f(x)$, if exists, is unique. (Pf: Exercise!)

Thm: "Sequential Criteria"

$$\lim_{x\to c} f(x) = L \iff \forall \text{ seq. } (x_n) \text{ in } A \text{ s.t. } (*)\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$$

we have $\lim(f(x_n)) = L$

Proof: "$\Rightarrow$" Let (x_n) be a seq in A s.t. $(*)$ holds

Let $\varepsilon > 0$ be fixed but arbitrary.

Since $\lim_{x\to c} f(x) = L$, $\exists\ \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad \begin{matrix} x \in A \\ 0 < |x - c| < \delta \end{matrix}$$

Since$^{(*)}$ $\lim (x_n) = c$, for the $\delta > 0$ above.

$\exists\ K = K(\delta) \in \mathbb{N}$ s.t. $0 \overset{(*)}{<} |x_n - c| < \delta \quad \forall n \geq K$

$\Rightarrow \quad |f(x_n) - L| < \varepsilon \quad \forall n \geq K$

"$\Leftarrow$" Suppose NOT, ie $\exists\ \varepsilon_0 > 0$ st $\forall\ \delta > 0$,

$\exists\ x_\delta \in A$ s.t. $0 < |x_\delta - c| < \delta$

BUT: $\quad |f(x_\delta) - L| \geq \varepsilon_0$

Take $\delta = \frac{1}{n}$, then get $x_n \in A$ s.t.

$$0 < |x_n - c| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - L| \geq \varepsilon_0 \quad \forall n \in \mathbb{N}$$

$\Rightarrow \quad \lim (x_n) = c \qquad$ BUT $\quad \lim(f(x_n)) \neq L$

$x_n \neq c \quad \forall n \in \mathbb{N}$

Contradiction!

$\blacksquare$

In summary, we have

<u>Setup</u>: $f : A \subseteq \mathbb{R} \to \mathbb{R}$, c : a cluster pt. of A (Note: not nec. belong to A)

<u>Def^n</u>: $\lim_{x \to c} f(x) = L \iff \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $x \in A$ and $0 < |x - c| < \delta$

<u>Sequential Criteria</u>

$\lim_{x \to c} f(x) = L \iff \forall$ seq. (x_n) in A s.t. $\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$

we have $\lim (f(x_n)) = L$

limit of function

limit of seq. of real numbers

<u>Remark</u>: This is helpful, in particular, to show that the limit $\lim_{x \to c} f(x)$ DOES NOT EXIST.

Taking the negation of Sequential Criteria above, we get:

<u>Cor 1</u>: f DOES NOT Converge to L as $x \to c$ $\iff \exists$ seq. (x_n) in A s.t. $\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$

BUT $(f(x_n)) \not\to L$

<u>Cor 2</u>: f "DIVERGES" as $x \to c$ (i.e. f DOES NOT Converge to L $\forall L \in \mathbb{R}$ as $x \to c$) $\iff \exists$ seq. (x_n) in A s.t. $\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim (x_n) = c \end{cases}$

BUT $(f(x_n))$ is divergent.

" Divergence Criteria "

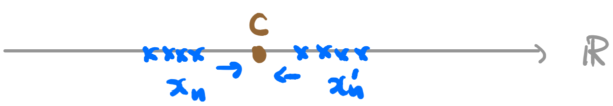
Proof of Cor. 2: "$\Leftarrow$" Easy.

"$\Rightarrow$" Argue by Contradiction. Assume f diverges at $x \to c$ but the R.H.S. fails to hold.

i.e. $\forall$ seq. (x_n) in A st. $(*)\begin{cases} x_n \neq c & \forall n \in \mathbb{N} \\ \lim(x_n) = c \end{cases}$

we have $\lim(f(x_n)) = L$ for some $L \in \mathbb{R}$

which may depend on the sequence (x_n)



Claim: The limit L DOES NOT depend on (x_n).

Pf of Claim: Suppose (x_n), (x_n') satisfy $(*)$, and

$$\lim(f(x_n)) = L \neq L' = \lim(f(x_n'))$$

Consider the new sequence

$$(y_n) := (x_1, x_1', x_2, x_2', x_3, x_3', \ldots\ldots)$$

satisfies $(*)$, then by hypothesis

$$(f(y_n)) := (f(x_1), f(x_1'), f(x_2), f(x_2'), \ldots\ldots)$$

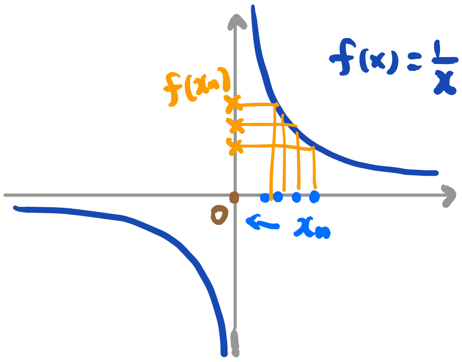
L $\quad L'$

is convergent, hence $L = L'$ $\qquad$ ▫

By sequential criteria, $\lim\limits_{x \to c} f(x) = L$ contradiction! $\qquad$ ▫

We now look at some examples where the limit of functions does not exist.

Example 1: $\lim_{x\to 0} \frac{1}{x}$ DOES NOT EXIST!



$f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$f(x) := \frac{1}{x}$

Pf: Take $(x_n) := \left(\frac{1}{n}\right)$.

Clearly, $\lim (x_n) = 0$, and

$A \ni x_n \neq 0 \quad \forall n \in \mathbb{N}$

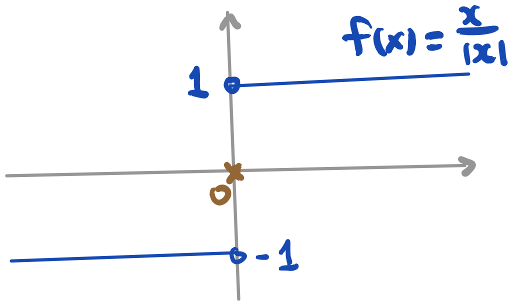
So $(*)$ is satisfied.

BUT $(f(x_n)) = (n)$ is DIVERGENT!

So we are done according to the divergence criteria above.

[Exercise: Prove directly using ε-δ def$^{\underline{n}}$ of limit.]

Example 2: $\lim_{x\to 0} \frac{x}{|x|}$ DOES NOT EXIST.



$f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$f(x) := \frac{x}{|x|}$

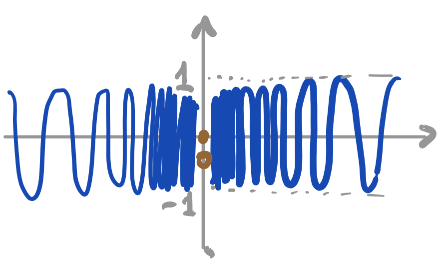
Pf: Take $(x_n) := \left(\frac{(-1)^n}{n}\right) \to 0$ ($\neq 0$)

satisfying $(*)$, the image seq.

$(f(x_n)) = ((-1)^n)$ DIVERGENT!

$\square$

Example 3: $\lim_{x\to 0} \sin\frac{1}{x}$ DOES NOT EXIST!



$f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$f(x) = \sin\frac{1}{x}$

Pf: Take $(x_n) := \left(\frac{1}{n\pi}\right) \to 0$ ($\neq 0$) BUT $(f(x_n)) = (0)$

Take $(x_n') := \left(\frac{1}{\frac{\pi}{2} + 2n\pi}\right) \to 0$ ($\neq 0$) BUT $(f(x_n')) = (1)$

So, let $(y_n) = (x_1, x_1', x_2, x_2', x_3, x_3', \dots) \to 0$ ($\neq 0$)

BUT $(f(y_n)) = (0, 1, 0, 1, 0, 1, \dots)$ DIVERGENT!

$\square$