MATH 2050C Mathematical Analysis I
2022-23 Term 2
Problem Set 3
due on Feb 10, 2023 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Gradescope on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.** All the exercises below are taken from the textbook.

Required Readings: Chapter 2.4, 2.5

Optional Readings: Chapter 2.5 on binary and decimal representations, (Rudin Appendix of Chapter 1) Dedekind construction of $\mathbb{R}$

Problems to hand in

Section 2.4: Exercise # 4(a), 6, 11, 19

Section 2.5: Exercise # 7

Suggested Exercises

Section 2.4: Exercise # 1, 2, 3, 4(b), 5, 7, 8, 9, 10, 12, 14, 15, 16

Section 2.5: Exercise # 2, 3, 8, 9, 10, 11

Challenging Exercises (optional)

1. (*Non-ordering of* $\mathbb{C}$) Prove that there is no ordering on $\mathbb{C}$ which makes it an ordered field.

2. (*Real exponential power*) Fix $b > 1$, show that one can define b^x for any $x \in \mathbb{R}$ as follow:

   (a) Let $r \in \mathbb{Q}$. Suppose $r = \frac{m}{n} = \frac{p}{q}$ such that $m, n, p, q \in \mathbb{Z}$, and $n, q > 0$. Prove that $(b^m)^{1/n} = (b^p)^{1/q}$. Hence b^r is well-defined for $r \in \mathbb{Q}$.

   (b) Prove that $b^r b^s = b^{r+s}$ for all $r, s \in \mathbb{Q}$.

   (c) Fix $x \in \mathbb{R}$. Define $B := \{b^t : t \in \mathbb{Q} \text{ and } t \leq x\}$. Prove that $b^x = \sup B$ when $x \in \mathbb{Q}$. Therefore, it makes sense to define $b^x = \sup B$ for any $x \in \mathbb{R}$.

   (d) Prove that $b^x b^y = b^{x+y}$ for any $x, y \in \mathbb{R}$.

3. (*Logarithm with base* b) Fix $b > 0$, $y > 0$, show that there is a unique $x \in \mathbb{R}$ such that $b^x = y$ by following the steps below:

   (a) For any $n \in \mathbb{N}$, show that $b^n - 1 \geq n(b-1)$. Hence, $b - 1 \geq n(b^{1/n} - 1)$.

   (b) If $t > 1$ and $n > \frac{b-1}{t-1}$, prove that $b^{1/n} < t$.

   (c) Suppose $w \in \mathbb{R}$ satisfies $b^w < y$. Show that $b^{w+\frac{1}{n}} < y$ for any sufficiently large $n \in \mathbb{N}$.

   (d) Suppose $w \in \mathbb{R}$ satisfies $b^w > y$. Show that $b^{w-\frac{1}{n}} > y$ for any sufficiently large $n \in \mathbb{N}$.

   (e) Define $A := \{w \in \mathbb{R} \ : \ b^w < y\}$. Show that $x := \sup A$ satisfies $b^x = y$. Prove that such an $x \in \mathbb{R}$ is unique.