1*. Let $f : (\mathbb{R} \setminus \{x_0\}) \to \mathbb{R}$ and $\delta_0 > 0$ s.t. $f(\cdot) < 0$ on $V_{\delta_0}(x_0) \setminus \{x_0\}$. Show that

$$\lim_{x \to x_0} f(x) = 0 \iff \lim_{x \to x_0} \left(\frac{1}{f(x)}\right) = -\infty$$

Discuss the corresponding results for $+\infty$ in place of $-\infty$.

2. Discuss the corresponding result for

(a) $x_0 = +\infty$

(b)* $x_0 = -\infty$

3*. Let $x_0 \in [-\infty, \infty]$ and $f, g : \mathbb{R} \setminus \{x_0\} \to \mathbb{R}$.

(a) Let $\lim_{x \to x_0} f(x) = 0$, and g be a bounded function. Show that $\lim_{x \to x_0} (f(x) g(x)) = 0$

Can the boundedness of g be dropped?

(b) Let $\lim_{x \to x_0} f(x) = \ell \in [-\infty, \infty]$ and $\lim_{x \to x_0} g(x) = \ell' \in [-\infty, \infty]$

Suppose $f(x) \leq g(x)$ $\forall x \neq x_0$. Show that $\ell \leq \ell'$.

(c). If $\lim_{x \to x_0} f(x) = \ell \in (-\infty, 0)$ and $\lim_{x \to x_0} g(x) = +\infty$, Show that $\lim\limits_{x \to x_0} \left(f(x) g(x) \right) = -\infty$

4.* Let $f : (0, \infty) \to \mathbb{R}$ be such that

$$\lim_{x \to \infty} \left(x f(x) \right) = \ell \in \mathbb{R}$$

Show that $\lim\limits_{x \to \infty} | x f(x) | = |\ell|$ and

$\lim\limits_{x \to \infty} f(x) = 0$.

5.(a) Let $f : (a, b) \to [m, M] \; (\subseteq \mathbb{R})$ be $\uparrow$.

Show that $\lim\limits_{x \to x_0-} f(x) = \sup \{ f(x) : x < x_0 \}$

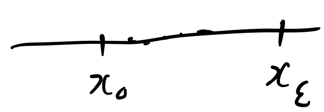
for all $x_0 \in (a, b]$, and

$\lim\limits_{x \to x_0+} f(x) = \inf \{ f(x) : x > x_0 \}$ $\forall$ $x_0 \in [a, b)$.

Set $\ell := \inf \{ f(x) : x > x_0 \}$ → 最大下界

(Hint for the last assertion: Let $\varepsilon > 0$. Then $\exists$ $x_\varepsilon > x_0$

s.t. $\ell + \varepsilon > f(x_\varepsilon)$. Use $\uparrow$ to get



$$f(x_\varepsilon) \geq f(x) \;\; \forall \, x \in (x_0, x_\varepsilon]$$

....)

Q5(b)*

Mimicing the sequential case, state and prove the Monotone Convergence theorem for functions on (a,b):

Let $f:(a,b)\to\mathbb{R}$ be monotone. Then

① $\lim_{x\to a+} f(x)$ exists in $\mathbb{R}$ iff ...... (填充)

② $\lim_{x\to b-} f(x)$ exists in $\mathbb{R}$ iff ----